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The Charles Stark Draper Laboratory, Inc.

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ACOSS ELEVEN (ACTIVE CONTROL OF SPACE STRUCTURES), Volume II

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The Active Control of Space Structures (ACOSS) Eleven contract includes three distinct areas of endeavor: Simulations Extension, HALO Optics and ACOSS. This report covers the work performed in each of these areas between May and November 1981. Simulations Extension and HALO Optics are presented in Volume I and ACOSS is presented in Volume II.

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LIST OF ACRONYMS

AVD	Air-Vehicle Detection
CSDL	The Charles Stark Draper Laboratory, Inc.
DARPA	Defense Advanced Research Projects Agency
DIS	Draper Integrated Simulations
DISCOS	Dynamic Interaction Simulation of Structures
DMA	Defense Mapping Agency
FFT	Fast Fourier Transform
GSS	Generic Scene Simulation
HAC	Hughes Aircraft Corporation
IFOV	Instantaneous Field of View
IMSL	International Mathematical and Statistical Libraries
IR	Infrared
LOS	Line of Sight
LS	Least Squares
lsc	Lower Semicontinuous Function
LSS	Large Space Structures
OTF	Optical Transfer Functions
PRA	Photon Research Associates
PSF	Point-Spread Function
TPBVP	Two-Point Boundary Value Problem
wlsc	Weakly Lower Semicontinuous Function
wslsc	Weakly Sequentially Lower Semicontinuous Function

PREFACE

Volume 2 of this report describes the work CSDL has done to investigate spacecraft control theory. Each of the six sections devoted to ACOSS reports on a different aspect of that work.

Section 4, "Compensated Truncation of Modal Models for Design of Control Systems," describes the selection process necessary in large space structure (LSS) control-system design using a truncated finite-element model. The truncated model must be selected properly and compensated explicitly for control and observation spillover, so the control system designed through this method can perform satisfactorily when implemented on the structure. Proper selection requires correct classification of structural modes into "primary" and "secondary" modes. Explicit compensation for truncation includes: placement of actuators and sensors, synthesis of the actuator and sensor influences once they are placed on the structure, and filtering of the actuator inputs and sensor outputs.

Section 5, "Ensuring Full-Order Closed-Loop Stability in the Reduced-Order Design of Output Feedback Controllers," builds on the studies performed during ACOSS 6 that established various conditions necessary to ensure full-order closed-loop asymptotic stability and robustness with reduced-order design of velocity and displacement output feedback controllers. Currently, the work in this area concentrates on how to apply such results to large flexible space structures and how to develop a reduced-order design technique that will ensure full-order closed-loop asymptotic stability.

The study includes preliminary development of computer-aided design software and acceleration output feedback control.

Section 6, "Design Freedom and the Implementation of Suboptimal Output Feedback Control," discusses the freedom inherent in design. The section states that often this freedom is sacrificed purposely when simplifying assumptions are made to avoid theoretical or computational difficulties. Since it is difficult to consider this topic without referring to specific applications, the section uses controller design as an example where work is being done to discover and exploit the freedom of choice in design. Then, the section uses suboptimal output feedback control as a case study which is relevant to ACOSS development.

Section 7, "Stochastic Output Feedback Compensators for Distributed Parameter Structural Models," presents recent progress on the stochastic output feedback design problem for distributed parameter plants. The results presented are an extension of work done under the previous contract.

The concepts developed are general enough to apply to a wide variety of fixed-form compensator design problems, and current studies are aimed at specializing the results to the optimal output feedback compensator design problem. The procedure developed will be applied to the design of velocity feedback controllers for a vibrating string. The results of this simple test should provide insight into the impact of various modeling assumptions on the convergence of the design procedure described.

Section 8, "Large-Angle Spacecraft Slewing Maneuvers," further develops work that was reported in the previous ACOSS contract. Specifically, the section presents techniques for improving the optimal torque profiles by allowing the solution process to determine the optimal terminal boundary conditions and by developing a control-rate penalty technique for producing smooth control profiles. Several example maneuvers are provided to demonstrate the practical application and utility of the techniques presented.

Section 9, "Order Reduction by Identification—Some Analytical Results," attempts to characterize control designs that will guarantee stability using a reduced-order model. This kind of design compromise is practiced regularly, but no one has verified the validity of such an approach.

The least squares (LS) method is used in this analysis because it is a relatively robust identification scheme and analytical expressions for order reduction already exist for it. The results of the analysis show that a reduced order controller can be built using the LS method of identification. It is planned to demonstrate the practicality of this approach on Draper Model #2 in the near future.

SECTION 4

COMPENSATED TRUNCATION OF MODAL MODELS FOR DESIGN OF CONTROL SYSTEMS

4.1 Introduction

Finite-element models of a large space structure are too large for the design of its control systems, let alone implementation of "modem" control systems in space. The model must be truncated substantially to reduce it to a reasonably low order. A proper truncation requires proper structural-mode classifications according to their influences on required performance and sensor outputs and their responses to probable disturbances and actuator inputs. The truncated modal models must explicitly be compensated for their truncation to prevent or reduce control spillover and observation spillover.

4.2 Reduced-Order Modeling for Control-System Designs by Modal Truncation

In practice, not all structural modes that are "molded" by a large finite-element model are to be calculated because of the increasing computational expense and inaccuracy. Yet, the number of modes commonly calculated (using NASTRAN, for example) are very large for a typical large flexible space structure. Only a very limited number of these calculated modes can be used to form a reduced-order model required for design of the structure's control systems, however. Some ideas for appropriate reduced-order modeling are being formulated here at CSDL. The following (here and Section 4.3) is a preliminary sketch.

Among the calculated modes, some are to be classified as "primary modes" and some as "secondary modes". Those modes that will influence the specified performance of the structure significantly (e.g., the line-of-sight error and defocus) and/or will be influenced significantly by probable disturbances on the structure (e.g., initial disturbances due to maneuvers, sinusoidal or random disturbances caused by on-board equipments) are classified as <u>primary</u> modes. Some primary modes are regarded as critical if a certain critical level is exceeded. With respect to a configuration of actuators and sensors placed on the structure, <u>secondary</u> modes are those <u>nonprimary</u> modes which <u>either</u> can be influenced significantly by the actuation <u>or</u> can influence the sensing significantly. Useful ranking techniques are being formulated and studied.

Naturally, those calculated modes which will not only be influenced significantly by the probable disturbances but also influence the specified performance significantly should be "modeled" (i.e., retained in the reduced-order model) for design of control systems. If possible, all other primary modes should be modeled as well. On the other hand, it is obvious intuitively that any calculated mode, be it primary or secondary, also should be modeled for design of control systems if it can be influenced strongly by the actuation or if it can influence the sensing strongly. A recent research into the linear-quadratic regulator design technique for application to large space structures has indicated that if strongly influential or strongly influenced secondary modes are not modeled, spillover can become a very serious problem. To sum up, some or all primary modes and certain secondary modes are to be modeled for design of control systems; all other modes are to be neglected.

4.3 Compensation for Truncation

Any reduced-order model carefully derived as such is still a coarse approximation of its original full-order model. The truncation must be adjusted and compensated first so that the closed loop performance of the resulting reduced-order model (with the control systems thus designed) can closely represent the closed-loop performance of the full-order model (with the same control systems). In principle, actuators on the structure should be configured so that influences on modeled primary modes are much stronger than on any other calculated modes and that the number of secondary modes is reduced to a minimum. Similarly, sensors also should be configured so that influences by modeled primary modes are much stronger than by any other calculated modes and that the number of secondary modes is reduced further. The first step in compensating for truncation is to adjust the configuration of the actuators and sensors properly, since an initial intuitive configuration generally is not proper with respect to such specific modeling requirements. It is ideal if control spillover and observation spillover can be prevented by proper placement of a proper number of actuators and sensors. The "placement step" in the three-step spillover reduction technique, which resulted from the ACOSS-4 study [4-1], can be used to generate insights of ideal locations and directions for actuators and sensors.

Because of practical constraints on the number, type, location, and direction of the actuators and sensors allowed on the structure, spillover may not be prevented completely. Assume that actuators and sensors have been placed on the structure and that the modeled modes for control design have been determined. Then, the second step in the compensation is synthesizing the influences of those actuators and sensors properly so that spillover concerning a judicious selection of secondary modes is prevented. The "synthesis step" of the aforementioned three-step technique is applicable. While ideas and methods for selecting such secondary modes are to be formulated, a preliminary development of the synthesis techniques has begun [4-2] and is being completed (see Section 4.4).

The third step in the compensation is to filter the actuator inputs and the sensor outputs so that control spillover and observation spillover of other unmodeled modes are reduced appropriately. Low-pass or band-stop filters are to be provided for proper attentuation of unwanted, spill-causing frequency components in the inputs and outputs, as was considered as the "filtering step" of the three-step spillover reduction technique. It is logical that control systems should be designed with these filters considered as integral parts of the reduced-order model, as was suggested earlier [4-1]. On the other hand, it also sounds reasonable intuitively that <u>posterior</u> attachment of low-pass filters to a control system which is designed using only the unfiltered reduced-order model could improve the full-order closed-loop performance of the control system. This common intuition is being examined (see Section 4.5).

4.4 A Completion of the Synthesis

Preventing control and observation spillover by using synthesizers has been discussed in References 4-1 and 4-2. A completion of the synthesis process is discussed in this section.

Once an adequate set of secondary modes has been determined to prevent spillover, the synthesizer, Γ, can be computed from Reference 4-2, Eq. (5-15).

$$\Gamma = Q_2 \hat{\Gamma} \tag{4-1}$$

where the synthesizer Γ is $m \times m'$, Q_2 is $m \times (m - \rho)$ and $\hat{\Gamma}$ is $(m - \rho) \times m'$. Here m is the number of (physical) actuators, m' is the number of columns chosen for Γ and represents the number of synthetic actuators, and ρ is the rank of the control influence matrix $\Phi_S^T B_F$ of such secondary modes.

The elements of matrix $\hat{\Gamma}$ are free parameters. The number m' of synthetic actuators can be arbitrary, but, without being redundant, $m - \rho$ can be considered an upper limit. To exercise the complete synthesis process, m' is chosen to be unity. So $\hat{\Gamma}$ is a vector of free parameters. Consider choosing $\hat{\Gamma}$ so that the synthesized primary control influences take on some desirable values (see Reference 4-2, Eq. (5-16)). Of course, Reference 4-2, Eq. (5-8) will always hold no matter what the choice for $\hat{\Gamma}$ is. Here $\hat{\Gamma}$ will be chosen to make the elements of the synthesized primary control influence vector all 1's, if possible. The relevant equation (Reference 4-2, Eq. (5-16)) is

$$\Phi_P^T B_F \Gamma = V_2 \hat{\Gamma} \tag{4-2}$$

where the primary control influence matrix $\Phi_P^T B_F$ is $p \times m$, p being the number of primary modes, and V_2 is $p \times (m - \rho)$. Note, however, that p is usually very much greater than $m - \rho$, so in general one cannot solve Eq. (4-2) for $\hat{\Gamma}$ such that $\Phi_P^T B_F \Gamma$ takes on the desired values exactly.

Consider using a linear least squares technique for finding $\hat{\Gamma}$ so that the error $||(\Phi_P^T B_F \Gamma)_{desired} - V_2 \hat{\Gamma}||$ is minimized. An International Mathematical and Statistical Libraries (IMSL) linear least-squares subroutine is used to determine a minimizing $\hat{\Gamma}$. A synthesizer Γ is then obtained using Eq. (4-1).

The complete synthesis approach to spillover prevention has been applied to Models 1 and 2. The results of post-multiplying the entire control influence matrix (secondary, primary, as well as tertiary parts) of both models are presented in the following.

Figure 4-1 illustrates the results for the example considered in Section 5.5.1 of Reference 4-2. Recall that the desired values for the elements of the synthesized primary control influence vector are all 1's. Here Modes 1 and 2 yielded poor synthesized control influence, while Modes 4 and 5 synthesized quite well. Interestingly, Modes 1 and 2 also have been shown to have a poor "modal degree of controllability". The synthesized tertiary influences were clearly not of the order of the secondary control influences (as was desired), but all were less than 1.4. These are indications of either poor placement of actuators or poor selection of secondary modes for spillover prevention.

$$
\left.\begin{matrix} 0.2775557561562891\text{D-}16 \\ 0.1665334536937735\text{D-}15 \\ 0.4163336342344337\text{D-}16 \\ -0.1387778780781446\text{D-}15 \end{matrix}\right\} \Phi_S^T B_F \Gamma
$$

$$
\left.\begin{matrix} 0.1633203111612015\text{D+}00 \\ -0.8500318695477063\text{D-}01 \\ 0.1121691406403187\text{D+}01 \\ 0.0972729391173360\text{D+}00 \end{matrix}\right\} \Phi_P^T B_F \Gamma
$$

$$
\left.\begin{matrix} 0.3315621655720254\text{D+}00 \\ 0.1303922005184048\text{D+}01 \\ 0.6155967078839501\text{D+}00 \\ 0.2749183336505013\text{D+}00 \end{matrix}\right\} \Phi_T^T B_F \Gamma
$$

Figure 4-1. Synthesized control influence matrix for Model 1.

This same synthesizer, Γ, was used to post multiply the control influence matrix of the perturbed Model 1. The result is shown by Figure 4-2, where $\tilde{\Phi}$ denotes the mode-shape matrix of the perturbed model. Some reduction was achieved in the secondary influences though the reduction was not uniform or adequate. The problem with poor placement or poor selection was compounded by parameter variation.

$$
\left.\begin{matrix} 0.355006693\text{D-}04 \\ 0.584594350\text{D-}02 \\ 0.995436440\text{D-}01 \end{matrix}\right\} \tilde{\Phi}_S^T B_F \Gamma
$$

$$
\left.\begin{matrix} 0.101549556\text{D+}00 \\ -0.645081959\text{D-}01 \\ 0.109292309\text{D+}01 \\ 0.701211710\text{D+}00 \end{matrix}\right\} \tilde{\Phi}_P^T B_F \Gamma
$$

$$
\left.\begin{matrix} 0.636934502\text{D+}00 \\ 0.141236880\text{D+}01 \\ 0.585492998\text{D+}00 \\ 0.223099504\text{D+}00 \end{matrix}\right\} \tilde{\Phi}_T^T B_F \Gamma
$$

Figure 4-2. Synthesized control influence matrix for perturbed Model 1.

This same synthesis process was also applied to Model 2 where the 78×19 elastic control influence matrix was to be synthesized. The synthesizer was computed from the results of the second synthesizer example in Reference 4-2, Section 5.5.2. The synthesized control influence matrix for Model 2 is shown in Figure 4-3. Notice that again some of the primary modes yielded synthesized values close to 1 while others did not. A number of the tertiary modes synthesized to small values as was desired. Note that the secondary modes chosen for spillover prevention were originally selected based only on knowledge of the 44 lowest frequency elastic modes and that the selection was based on a high "modal degree of controllability". In the synthesized secondary partition, the third mode (mode nine) was several orders of magnitude above the other secondary modes. This is because mode nine corresponds to the row where the potential pivots were too small to eliminate in the secondary partition of the control influence matrix (see Reference 4-2, Figure 5-3).

$$\left.\begin{array}{r}
-0.56833344605809790\text{-}16 \\
0.16859400328853810\text{-}16 \\
0.50487650986766380\text{-}06 \\
0.55511151231257830\text{-}16 \\
-0.20556473190325160\text{-}15 \\
0.14040418133687280\text{-}15 \\
0.90205620750793970\text{-}16 \\
0.77195194680967920\text{-}16 \\
0.10581813203458520\text{-}15 \\
0.47046327472871980\text{-}17 \\
0.25641381379282180\text{-}16 \\
-0.31770088269140670\text{-}17 \\
0.67654215563095480\text{-}16 \\
0.22741114098103850\text{-}17 \\
-0.49385408956714730\text{-}16 \\
-0.62670415652019710\text{-}18 \\
0.13030754860560160\text{-}16
\end{array}\right\} \Phi_S^T B_F \Gamma$$

$$\left.\begin{array}{r}
-0.66365441463864340\text{-}04 \\
-0.91650539479413670\text{-}04 \\
0.88146065284821370\text{-}04 \\
0.17134221161214800\text{-}01 \\
0.21425075226766790\text{-}03 \\
-0.11176832998884880\text{-}02 \\
0.10815794392352760\text{-}03 \\
0.14056562033595170{+}00 \\
0.39809583433409310\text{-}03 \\
-0.28481079329020960\text{-}02 \\
0.55984482946594060\text{-}01 \\
0.74861138570800980\text{-}01 \\
0.10613944472106150{+}01 \\
0.12763570757435780{+}01 \\
-0.22809258464196020{+}00 \\
0.32023639482424540{+}00 \\
0.66156169496378750{+}00
\end{array}\right\} \Phi_P^T B_F \Gamma$$

$$\left.\begin{array}{r}
0.69879179462339040\text{-}05 \\
0.82825521097245640\text{-}04 \\
0.84415802961035580\text{-}05 \\
-0.21549715789466210\text{-}13 \\
0.10320388668420850\text{-}02 \\
-0.18352399061835150\text{-}13 \\
-0.61577062542226910\text{-}14 \\
0.31941945464432280\text{-}04 \\
0.15886816432557170\text{-}03 \\
0.99519636317025920\text{-}14 \\
-0.14012405454371860{+}01 \\
-0.40843852846481010{+}00 \\
-0.66295932243994120{+}00 \\
0.90568049382866150{+}00 \\
-0.92652865443553160\text{-}03 \\
-0.84569965213267440{+}00 \\
-0.14267702304014540{+}00 \\
-0.31831436522087720{+}00 \\
-0.49519919170249040{+}00 \\
0.18154617373606430{+}00 \\
-0.10373808703545270{+}01 \\
-0.45270326178271640{+}00 \\
0.43589688530672810{+}00 \\
0.96500754593004900{+}00 \\
0.27591205733934130\text{-}01 \\
-0.39589120212958600{+}00 \\
0.93318180729178530{+}00 \\
-0.46660381025905000{+}00 \\
0.74877197056352230\text{-}13 \\
0.20569142386744490\text{-}05 \\
-0.11896964606033670{+}01 \\
-0.74893330163065200{+}00 \\
0.25202580940456640\text{-}14 \\
-0.43856180084186660\text{-}05 \\
0.28542001632795950{+}01 \\
-0.90594557045505440{+}01 \\
-0.24318932033967380{+}02 \\
0.75304370848182970{+}01 \\
-0.27639688971224770{+}01 \\
-0.28592104092137920{+}01 \\
-0.94127130271942650{+}01 \\
0.21074405092789800{+}02 \\
0.12537135442541590\text{-}01 \\
0.23405268262630910{+}01
\end{array}\right\} \Phi_T^T B_F \Gamma$$

Figure 4-3. Synthesized control influence matrix for Model 2.

4.5 Filters for Spillover Reduction

The reduction of spillover by filtering the actuator inputs and sensor outputs appropriately was originally discussed in [4-1]. The general effects of the filters on the transient and steady-state responses have been investigated and are reported briefly here. Some preliminary results on posterior addition of filters to Model 1 are discussed also.

One may consider using a filter to reduce control spillover. Then one is concerned with the effect of the filter on steady-state response and transient response with a general sinusoidal input to a very lightly damped system. Two simple filters are being examined: first- and second-order low-pass filters of the following kind

$$G_c(s) = \frac{1}{\tau s + 1} \tag{4-3}$$

$$G_c(s) = \frac{1}{s^2 + ds + c^2} \tag{4-4}$$

The time response (magnitude and phase angle) of a mode to an input where filtering is included has been determined. The symbol manipulation language, MACSYMA, was useful for some of this work. The input-frequency-dependent effect of the filters on the steady-state responses is as well known. On the other hand, the filters reduce the magnitude and phase angle of the transient response of each mode uniformly for all input frequencies. The reduction, of course, varies with the natural frequency of the mode.

The common intuition of posterior attachment of low-pass filters to vibration controllers was studied as a means of reducing control spillover to high-frequency modes using Model 1. A modal dashpot design of velocity output feedback controllers (based on critical Modes 1, 2, 4, and 5) was considered because of its commonly recognized robustness in closed-loop stability [4-2], [4-3]. CSDL found that adding a second-order low-pass filter, with its natural frequency lying between Modes 5 and 6, would destabilize the 12-mode closed-loop system; intuitively such a filter was reasonable for reducing control spillover to Modes 6 and beyond. It was found also that adding a filter with a much higher natural frequency (e.g., between Modes 8 and 9) would not cause closed-loop stability.

Specifically, the damping ratio to be achieved for the primary modes with this output feedback controller is 0.1. Two cases are considered. In the first case, the feedback gain matrix is

$$G_V = -(\Phi_P^T B_F)^{\#}(2Z_P\Omega_P)(C_V\Phi_P)^{\#} \tag{4-5}$$

and the control u is

$$u = G_V y \tag{4-6}$$

The six actuators and the six sensors are colocated, respectively. In the second case, the six (physical) actuators are combined into four synthetic actuators so that all spillover is prevented to modes three and six. The synthesizer Γ, which is (6 × 4), is chosen so that the synthesized primary-mode control influence

$$\Phi_P^T B_F \Gamma$$

is the identity matrix. The six (physical) sensors are also synthesized in the same way but "dualized". So, for the four pairs of synthetic actuators and synthetic sensors, the feedback gain simply becomes

$$G_V = -2Z_P \Omega_P \tag{4-7}$$

The intuitively positive effects of filtering both actuator inputs and sensor outputs (toward reducing closed-loop spillover and thus allowing the damping ratio to be achieved by the primary modes) will be demonstrated in a later report.

For now, the potentially negative effects of filter insertion on system stability is investigated. (Low-pass filters also could represent the dynamics of the actuators.) The first-order filter above was considered for these values of τ: 0.05, 0.222, 0.25, 0.2, 1.0, and 10. The system remained stable for this range of τ in both cases, and the closed-loop poles of the 12-mode model (plus filters) moved toward the $j\omega$ axis as τ increased. The stability of the system was not preserved when the second-order filter was inserted. Two values of c were chosen ($c = 4$ and $c = 6$), and d was chosen to make the filter critically damped. When c was 4, the closed-loop system became unstable in both cases. With $c = 6$, however, the system remained stable in both cases.

4.6 Conclusion

Truncated modal models of a large space structure need to be selected properly and compensated explicitly for control spillover and observation spillover, so that the control systems thereby designed can perform satisfactorily when implemented on the structure. Proper selection requires proper classification of structural modes into "primary" and "secondary" modes. Explicit compensation for truncation includes

(1) Proper placement of actuators and sensors.

(2) Proper synthesis of the influences of the actuators and the sensors once they are placed on the structure.

(3) Proper filtering of the actuator inputs and sensor outputs.

Direct applications of state-of-the-art design techniques will then become possible, and the resulting designs more effective in closed-loop performance.

LIST OF REFERENCES

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4-2. Active Control of Space Structures—Final Report, CSDL Report R-1454, February 1981.

4-3. Actively Controlled Structures Theory—Interim Technical Report, Vol. 1, Theory of Design Methods, CSDL Report R-1249, April 1979.

SECTION 5

ENSURING FULL-ORDER CLOSED-LOOP STABILITY IN THE REDUCED-ORDER DESIGN OF OUTPUT FEEDBACK CONTROLLERS

5.1 Introduction

As a result of ACOSS-6 studies, CSDL has established various conditions for ensuring full-order closed-loop asymptotic stability and robustness with reduced-order design of velocity and displacement output feedback controllers; for details, see [5-1] Section 3. The emphasis of the current studies is to apply such results to large flexible space structures and develop a reduced-order design technique that will ensure full-order closed-loop asymptotic stability.

The study includes a preliminary development of computer-aided design software. Acceleration output feedback control is included in this development also. The full-order closed-loop asymptotic stability conditions are therefore extended to cover the reduced-order design of acceleration output feedback controllers.

Some important theoretical or technical issues that need to be addressed in depth in later work are discussed briefly at the end of this section.

5.2 Problem Formulation

Consider the following finite-element representation of large space structures

$$M\ddot{q} + D\dot{q} + Kq = f \tag{5-1}$$

where vector q denotes the L generalized coordinates, vector f the L generalized forces; matrices M, D, and K denote the mass (or inertia), the damping, and the stiffness, respectively. As usual, M is real, symmetric, and <u>positive definite</u>, while both D and K are real, symmetric, and <u>nonnegative definite</u>. Let there be m force (or torque) actuators for control of structural motions

$$f = B_F u \tag{5-2}$$

where vector u denotes the m actuator inputs, one for each actuator, and $L \times m$ matrix B_F denotes the actuator influence coefficients. Also let there be ℓ_A acceleration sensors, ℓ_V velocity sensors, and ℓ_D displacement sensors for measurement of structural motions

$$\begin{cases} y_A = C_A\ddot{q} \\ y_V = C_V\dot{q} \\ y_D = C_D q \end{cases} \tag{5-3}$$

where vector y_A denotes the ℓ_A acceleration-sensor outputs, vector y_V the ℓ_V velocity-sensor outputs, and y_D the ℓ_D displacement-sensor outputs; the $\ell_A \times L$ matrix C_A, the $\ell_V \times L$ matrix C_V, and the $\ell_D \times L$ matrix C_D denote the influence coefficients of the acceleration, velocity, and displacement sensors, respectively.

Now, consider the following form of output feedback control

$$\begin{cases} u_A = -G_A y_A \\ u_V = -G_V y_V \\ u_D = -G_D y_D \end{cases} \tag{5-4}$$

where the $m \times \ell_A$ matrix G_A, the $m \times \ell_V$ matrix G_V, and the $m \times \ell_D$ matrix G_D denote the feedback gains.

Partitioning the actuator-influence matrix appropriately yields the following expression of the control force applied to the structure

$$B_F u = [B_A \mid B_V \mid B_D] \begin{bmatrix} u_A \\ \hline u_V \\ \hline u_D \end{bmatrix} = B_A u_A + B_V u_V + B_D u_D \tag{5-5}$$

Substituting Eq. (5-2) through (5-4) in Eq. (5-1) yields the following expression for the closed-loop system

$$(M + B_A G_A C_A)\ddot{q} + (D + B_V G_V C_V)\dot{q} + (K + B_D G_D C_D)q = 0 \tag{5-6}$$

It follows that this output-feedback control results in an alteration of the mass, damping, and stiffness properties of the structure. The feedback gains G_A, G_V, and G_D can be designed using any technique for satisfying any specified performance requirements, but the asymptotic stability of the resulting closed-loop system must be ensured, particularly for large precision space structures.

A common practical condition for asymptotic stability is that the augmented mass matrix $(M + B_A G_A C_A)$, the augmented damping matrix $(D + B_V G_V C_V)$, and the augmented stiffness matrix $(K + B_D G_D C_D)$ are all positive definite. Symmetry of both the mass matrix $(M + B_A G_A C_A)$ and the stiffness matrix $(K + B_D G_D C_D)$ are always assumed in the literature. Such positive-definite conditions are not satisfied easily for large space structures since matrix products $B_A G_A C_A$, $B_V G_V C_V$, and $B_D G_D C_D$ practically can never be made positive definite. Moreover, the design of control systems for large space structures commonly is based on, not the whole model, Eq. (5-1), but a truncated version. Thus, full-order closed-loop asymptotic stability is even more difficult to ensure without any special efforts in the reduced-order design.

Expressed in modal coordinates, the full-order model, Eq. (5-1) and (5-3), become

$$\ddot{\eta} + \Delta\dot{\eta} + \Sigma\eta = \Phi^T f \tag{5-7}$$

$$\begin{cases} y_A = C_A\Phi\ddot{\eta} \\ y_V = C_V\Phi\dot{\eta} \\ y_D = C_D\Phi\eta \end{cases} \tag{5-8}$$

where

$$\Phi\eta = q \tag{5-9}$$

$$\Phi^T M\Phi = I \tag{5-10}$$

$$\Phi^T D\Phi = \Delta \tag{5-11}$$

$$\Phi^T K\Phi = \Sigma \tag{5-12}$$

To enable the design of a control system, this modal model is commonly truncated to reduce its order. Let the undamped natural modes retained be denoted by subscipt M Then, the design of feedback gains G_A, G_V, and G_D is based only on the following reduced-order model

$$\ddot{\eta}_M + \Delta_M\dot{\eta}_M + \Sigma_M\eta_M = \Phi_M^T f \tag{5-13}$$

$$\begin{cases} y_A = C_A\Phi M\dot{\eta}M \\ y_V = C_V\Phi_M\dot{\eta}_M \\ y_D = C_D\Phi_M\eta_M \end{cases} \tag{5-14}$$

The problem is to ensure that the resulting full-order closed-loop system, given by Eq. (5-6) or equivalently by Eq. (5-2), (5-4), (5-5), and (5-8) combined with Eq. (5-7), is asymptotically stable while the feedback gains G_A, G_V, and G_D are being designed using a reduced-order model such as given by Eq. (5-13) and (5-14). For comments on the underlying difficulties in ensuring full-order closed-loop asymptotic stability with reduced-order design of velocity and displacement output feedback control, see Reference 5-1, Section 3. Those comments also apply to reduced-order design of acceleration output feedback control.

5.3 Full-Order Closed-Loop Asymptotic Stability Conditions

The stability conditions previously established for reduced-order design of velocity and displacement output feedback can easily be extended to cover the design of acceleration output feedback. Let the modeled modes contain those that can influence the specified performance (say, line-of-sight accuracy) of the structure and/or the sensor outputs significantly and those that can be influenced significantly by disturbance on the structure and/or the actuator inputs. Assume without loss of generality that all rigid modes are modeled modes and that all unmodeled modes are elastic modes, each of which has some positive amount of inherent damping. Following the same development as [5-1] Section 3.3, the following conditions can be stated.

The full-order closed-loop system given by Eq. (5-6) is asymptotically stable if the following three conditions are all satisfied.

(1) The acceleration output feedback gain G_A ensures that the product $B_A G_A C_A$ is both symmetric and nonnegative definite.

(2) The velocity output feedback gain G_V ensures that $\Delta_M + \Phi_M^T B_V G_V C_V \Phi_M$ is positive definite and $B_V G_V C_V$ is nonnegative definite.

(3) The displacement output feedback gain G_D ensures that $\Sigma_M + \Phi_M^T B_D G_D C_D \Phi_M$ is positive definite and $B_D G_D C_D$ is both symmetric and nonnegative definite.

A few remarks are necessary. Condition (1) ensures both the symmetry and the positive definiteness of the augmented mass matrix $(M + B_A G_A C_A)$. Condition (2) ensures the positive definiteness of the augmented damping matrix $(D + B_V G_V C_V)$. Condition (3) ensures both the symmetry and the positive definiteness of the augmented stiffness matrix $(K + B_D G_D C_D)$.

5.4 Principle of the Reduced-Order Design Technique

Condition (3) takes on the most general form and is the strongest among the three conditions. The discussions on the design technique will focus on ensuring Condition (3) in the reduced-order design of displacement output feedback control. Thus, the following reduced-order design technique also can be used or adapted for reduced-order design of velocity and acceleration output feedback control.

Step 1: Let Σ_M^* be a nonnegative definite matrix such that

$$\bar{\Sigma}_M \stackrel{d}{=} \Sigma_M + \Sigma_M^* \tag{5-15}$$

is a positive definite matrix representing desirable closed-loop stiffness of the modeled modes, and set

$$\Phi_M^T B_D G_D C_D \Phi_M = \Sigma_M^* \tag{5-16}$$

This ensures that $\Sigma_M + \Phi_M^T B_D G_D C_D \Phi_M$ is positive definite and $B_D G_D C_D$ is nonnegative definite.

Step 2: Let the feedback gain G_D be determined as a product of three matrices:

$$G_D = L_D S_D R_D \tag{5-17}$$

To ensure the symmetry of $B_D G_D C_D$, set

$$B_D L_D = (R_D C_D)^T \tag{5-18}$$

and restrict Σ_M^* to be a symmetric matrix.

Step 3: Solve for L_D, S_D, and R_D from the following matrix equations

$$\left[-B_D \;\middle|\; C_D^T\right] \left[\begin{array}{c} L_D \\ \hline R_D^T \end{array}\right] = 0 \tag{5-19}$$

$$\Phi_M^T B_D L_D S_D R_D C_D \Phi_M = \Sigma_M^* \tag{5-20}$$

5.5 Discussions

Matrix Eq. (5-19) can be rewritten as

$$AX = 0 \tag{5-21}$$

with A representing the known part

$$\left[-B_D \;\middle|\; C_D^T\right]$$

and X the unknown part

$$\left[\begin{array}{c} L_D \\ \hline R_D^T \end{array}\right].$$

It is now of the same form as the one in the synthesis of actuator influences for prevention of control spillover. The analytical expression of the general closed-form solution already has been given in Eq. (2-30) and (2-31) of Reference 5-2, Section 2, and that expression can be used directly. A preliminary computer program, similar to that used in the synthesis of actuator influences, has been coded in PL/I; a Gaussian elimination method is used.

Similarly, Eq. (5-20) can be rewritten as

$$AXB = C \tag{5-22}$$

with A, B, and C representing the known parts $(\Phi_M^T B_D L_D)$, $(R_D C_D \Phi_M)$, and Σ_M^* respectively while X represents the unknown part S_D. Note that L_D and R_D become known if Eq. (5-19) is solved first. The analytical expression of the general closed-form solution recently reported in Reference 5-1, Section 3.7.5 can also be used directly. A preliminary computer program using a Gaussian elimination method has also been coded.

A preliminary combination of these two programs to execute Step 3 of the design technique has been completed and debugged with Model No. 1. To further develop this design technique, several theoretical or technical issues need to be addressed in depth. The general solution for Eq. (5-19) contains a matrix of free parameters. A first issue to be addressed is how to handle such free parameters: Is it useful at all to retain the free parameters in matrices L_D and R_D? What are the tradeoffs if the free parameters are passed over to matrix S_D?

Both Eq. (5-19) and (5-20) require performing column and row operations on the known matrices. A second issue is the impact of different operation selections on the feedback gains.

At the end of Step 3, Condition(3) is ensured, but the solution S_D, and hence the feedback gain G_D, is in general a function of free parameters in the general solution of Eq. (5-20). The use of a general solution (instead of a pseudoinverse solution) is advantageous because it preserves the available free parameters. A third issue is how to use these free parameters to satisfy specified performance after the appropriate stability conditions have been ensured.

The design technique presented in this section does not require collocation of actuators with sensors or the symmetry of feedback gain matrix G_D, unlike common practice. Positive definiteness of gain G_D is not required either.

When this reduced-order design technique and the appropriate computer-aided design software are developed satisfactorily with respect to Model No. 1, CSDL will demonstrate their application to the much more complex Model No. 2.

LIST OF REFERENCES

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5-2. Actively Controlled Structures Theory Final Report, CSDL Report R-1338, December 1979.

SECTION 6

DESIGN FREEDOM AND THE IMPLEMENTATION OF SUBOPTIMAL OUTPUT FEEDBACK CONTROL

6.1 Introduction

The principal focus in this section is on the notion of recognizing, characterizing, and exploiting to advantage the freedom of choice inherently available in design. Although the specifics discussed relate to the design of feedback controllers for linear multivariable systems, such a notion clearly has much wider applicability. It is unfortunate that potential freedom in the design process sometimes remains unrecognized, much like treasure hidden in a field waiting to be discovered. More often, perhaps, such freedom is purposely sacrificed by making simplifying assumptions to avoid dealing with certain theoretical or computational difficulties that appear unpleasant and seemingly insurmountable. However, when sufficient motivation exists, the effort expended in facing the difficulties squarely often provides insight that leads to a much deeper understanding of the design problem and which reveals a wider class of possible solutions [6-1]. Such motivation certainly exists in designing active feedback controllers for vibration suppression in large flexible space structures when the precision required in pointing and tracking with the structure exceeds the present capability. Performance improvements associated with tapping previously unexploited design freedom could be significant.

The potential for design freedom often arises in connection with singularities that are allowed to remain in the mathematical model of the system being studied. Hence, it is difficult to make ideas on this matter precise apart from specific applications. In Section 6.2, a number of specific recent attempts to discover and exploit the freedom of choice in controller design are discussed to bring out some important general ideas. Special attention is then given, in Section 6.3, to a particular design approach (suboptimal output feedback control [6-2]) that has relevance in large flexible structure applications which has been enhanced significantly by exposing previously hidden potential for design freedom. The current status of efforts to exploit this freedom to advantage is described.

6.2 Design Freedom

For clarity of discussion, some notation is developed briefly. Consider the standard modal representation for a flexible structure

$$\ddot{\eta} + 2Z\Omega\dot{\eta} + \Omega^2\eta = (\Phi^T B_F)u$$

$$y = (C_p\Phi)\eta + (C_v\Phi)\dot{\eta}$$

where

$\eta \equiv (\eta_1,\ldots,\eta_n)^T$ is the vector of modal coordinates

Ω: $n \times n$ is the diagonal matrix containing the modal frequencies $\omega_1,\ldots,\omega_n$

Z: $n \times n$ is the diagonal matrix containing the modal damping ratios $\xi_1,\ldots,\xi_n$ associated with inherent material damping

Φ: $n \times n$ is the matrix with columns that are the structural mode shapes $\phi^1,\ldots,\phi^n$

B_F: $n \times m$ is the static gain matrix of the actuators driven by the control $u \equiv (u_1,\ldots,u_m)^T$

C_P: $\ell \times n$ and C_V: $\ell \times n$ are the static gain matrices of the displacement and rate sensors, respectively

$y \equiv (y_1,\ldots,y_\ell)^T$ is the system output

Superscript "T" denotes matrix transpose

Several recent investigations into controller design for large flexible space structures have uncovered substantial potential for improved system performance through design freedom which was made available by relaxing restrictions on the rank of certain parameter matrices formed from $\Phi^T B_F$ and $(C_P,C_V)\Phi$ through mode selection. These studies are described as follows:

(1) Study of a suboptimal output feedback approach [6-2] revealed that the equation for the feedback gain matrix has an infinite number of exact solutions when a reduced order observation matrix associated with $(C_P,C_V)\Phi$ has less than full rank [6-3]. Specific indications of potential benefits to be gained by exploiting this nonuniqueness were given.

(2) A study of techniques for alleviating control and observation spillover associated with reduced-order controllers [6-4] through the placement of actuators or the synthesis of their influences revealed that spillover to certain classes of modes can be prevented if a reduced-order matrix associated with $\Phi^T B_F$ is allowed to have less than full column rank.

(3) In a study of damping and stiffness augmentation through output feedback [6-5], it was observed that the equation for the feedback gain matrix that realizes prespecified closed-loop damping (Δ) and stiffness (K) matrices has an infinite number of solutions when reduced-order matrices associated with $\Phi^T B_F$ and $(C_P, C_V)\Phi$ are allowed to have less than full rank. Specific indications of potential benefits from such freedom of choice are given. The relaxation of the usual assumptions on the closed-loop matrices Δ and K also contributes to the freedom of choice.

Similar investigations of freedom in estimator design have been made [6-6].

Freedom of choice in design can be of the most value when particular aspects of it can be characterized and systematic procedures for effectively exploiting it can be developed. This point is worth clarifying by describing a specific and significant result of this nature in some detail. Consider the capabilities of full-state feedback in a linear time-invariant multivariable system

$$\dot{x} = Ax + Bu \; ; \quad u = Kx + L\eta \tag{6-1}$$

$$y = Cx \tag{6-2}$$

where

$$x \equiv (x_1, \ldots, x_n)^T$$

$$u \equiv (u_1, \ldots, u_m)^T$$

$$y \equiv (y_1, \ldots, y_\ell)^T$$

(η represents an external input). The property of controllability [6-7] has been characterized in the frequency domain as the ability to find a matrix K such that the poles of the closed-loop system matrix A + BK formed from Eq. (6-1) coincide with an arbitrary preassigned symmetric set of complex numbers [6-8]. It is easy to show, using the companion canonical form [6-9] for controllable systems, that if there is only one input ($m = 1$), specification of all the (n) closed-loop eigenvalues determines the (n) elements of an associated matrix K uniquely. This uniqueness disappears when more than one input is present, which leaves open the possibility of choosing the (nm) elements of the feedback gain matrix to achieve desirable properties of the closed-loop system beyond the assignment of eigenvalues. The problem of characterizing this freedom of choice was recognized as a significant open question quite recently [6-10].

A very elegant characterization of this freedom was provided subsequently by Moore [6-11] and Klein [6-12] (an amplified version of [6-11] can be found in [6-13]). In essence, this characterization states that the freedom beyond the ability to assign eigenvalues is the ability to assign associated eigenvectors. Although such eigenvectors cannot be selected arbitrarily, the restrictions upon the class from which the eigenvectors must be taken are quite mild. A precise statement of this result (Theorem 6-2) is given in Section 6.6.1.

Once such a characterization is established, attention is focused on discovering the most effective means for exploiting the indicated freedom. Whereas the assignment of eigenvalues determines the speed (i.e., natural frequencies, damping ratios) of the closed-loop dynamics, the assignment of eigenvectors determines the shape of those dynamics. Expressing the general solution for the closed-loop dynamics in spectral form which explicitly displays the eigenvalues and eigenvectors provides some insight into possible approaches for selecting candidate eigenvectors [6-13]. However, considerable ingenuity is needed to find the most advantageous selections; a number of different approaches have been proposed [6-11, 6-12, 6-13, 6-14, 6-15, 6-16, 6-17], and several of them proceed so as to reduce the sensitivity of the eigenvalues and eigenvectors to perturbations in the closed-loop system matrix. Once a set of eigenvectors satisfying the requirements of Theorem 6-2 has been selected, the corresponding gain matrix can be determined readily by following the constructive procedure given in the proof of the theorem [6-13]. In Section 6.6.2, a few detailed remarks regarding the art of selecting appropriate eigenvectors are given.

6.3 Case Study: Numerical Implementation of Suboptimal Output Feedback Control

6.3.1 Background

When the Kosut approach to suboptimal output feedback control [6-2] was reexamined recently to investigate its applicability to reduced-order controller design for flexible spacecraft [6-18], the potential for substantial freedom of choice in gain matrix selection came to light. Theoretical development characterizing this freedom and examples illustrating the potential for exploiting this freedom are given in [6-3]. The essential observation that generates the added design freedom is the algebraic consistency of a linear equation for determining the output feedback gain matrix, even when the coefficient matrix is rank-deficient. In such cases (i.e., rank deficiency), a whole family of exact solutions for the feedback gain matrix exists; parameterization of this family may often be selected to improve the performance of the full-order system which is driven by a reduced-order controller.

However, for this design approach to be feasible with large-order systems, a reliable computational procedure for dealing with rank-deficient linear algebraic equations is needed. Solution of such equations is a very delicate numerical problem; useless results are virtually certain if traditional solution procedures (e.g., Gaussian elimination) are used [6-19; pp. 200-218]. In this section, a mature version of a computational procedure initially outlined in [6-20] for obtaining the general solution of rank-deficient linear algebraic equations is presented. This procedure takes advantage of the special structure associated with the underlying controller design problem and expresses the solution in a form particularly appropriate for using the design freedom inherent in the extended Kosut approach [6-21].

It should be observed that the difficulty in solving rank-deficient linear equations is not algebraic; the theory is classical [6-22; pp. 73-109]. Rather, the difficulty is due to inevitable rounding errors that occur in actual computations which make the numerical determination of matrix rank extremely difficult [6-23]. Considerable work has been done on this problem, but very little of it has been reported in standard numerical analysis textbooks [6-24, 6-25]. Attempts to extend Gaussian elimination methods to treat rank-deficient equations have not proved very successful, especially when an accurate determination of rank is required [6-26, 6-27]. Considerably more success has accompanied development of iterative solution techniques, most of which are designed specifically for use in problems with sparse coefficient matrices [6-28, 6-29, 6-30, 6-31, 6-32]. The most successful development to date has been a direct (i.e., noniterative) technique based on an algorithm of Golub and Kahan [6-33] that employs a decomposition of the coefficient matrix to display its singular values [6-34]. This technique has been published as an ALGOL procedure [6-35] and incorporated into widely available robust mathematical software [6-36]. It is widely recognized as the only reliable approach to problems in which an accurate determination of numerical rank is essential [6-23, 6-34]. A precise development of what "numerical rank" really means has appeared recently [6-37].

For the present application, precise determination of the rank of the coefficient matrix in the gain equation is very important since the number of free parameters in the general solution for the gain matrix is proportional to the rank deficiency of the coefficient matrix [6-3]. Moreover, the structure of the underlying control design problem suggests that the coefficient matrix generally will be dense rather than sparse. Therefore, the procedure for numerical solution of the Kosut gain equation is built around the singular value decomposition.

6.3.2 Problem Statement

The algebraic equation to be solved numerically has the form

$$AX = B \tag{6-3}$$

With reference to the underlying problem of reduced-order controller design [6-3], matrices A: $\ell \times \ell$ and B: $\ell \times m$ have the structure

$$A^T = C_C P C_C^T , \quad B^T = F^* P C_C^T \tag{6-4}$$

where

C_C: $\ell \times 2N$ is the state-to-output observation matrix in a reduced-order plant having 2N states, m inputs, and ℓ outputs

P: $2N \times 2N$ is the symmetric positive-definite solution of a Liapunov matrix equation associated with the stable closed-loop system matrix of a reference plant of order 2N

F^*: $m \times \ell$ is the full-state feedback gain matrix in that reference plant

The gain matrix to be determined is $G = X^T$: $m \times \ell$.

It follows directly from the structure of A and B (Eq. (6-3) and (6-4)) that the gain equation, Eq. (6-3), is always algebraically consistent, regardless of the rank of A (= rank (C_C)) [6-3]. In other words, the best least-squares approximation to a solution of Eq. (6-3) is, in fact, an exact solution. When the coefficient matrix A is rank deficient, say rank (A) = $r < \ell$, the general solution of Eq. (6-3) contains $m \cdot (\ell - r)$ arbitrary parameters [6-3]. The problem treated is to develop a procedure for computing the general solution to Eq. (6-3). The procedure is designed so that the free parameters are displayed explicitly in a manner that facilitates their selection for improving the performance of the full-order (finite-dimensional) plant driven by the reduced-order controller in the underlying control design problem. Before this is described, a few essential facts about generalized inverses and the singular value decomposition are reviewed.

6.4 Some Facts from Linear Algebra

6.4.1 Generalized Inverses

A matrix is invertible if it is square and nonsingular. Otherwise, in general, some restriction of its domain (viewing the matrix as a mapping) is required before any property of invertibility can be achieved. The notion of a generalized inverse makes the appropriate restrictions and the corresponding invertibility properties precise.

The Moore-Penrose inverse of a real matrix A: $\mu \times \nu$ is the unique solution of the equations [6-38, 6-39]

$$AXA = A \tag{6-5}$$

$$XAX = X \tag{6-6}$$

$$(AX)^T = AX \tag{6-7}$$

$$(XA)^T = XA \tag{6-8}$$

and is denoted by $A^\dagger: \nu \times \mu$. It is identical with the usual inverse of A when A is square and nonsingular. A key to understanding this notion is the geometric interpretation of the composite maps $A^\dagger A$ and $AA^\dagger$ as projections. In particular [6-38]

$$A^\dagger A \text{ is a projection onto } R(A^\dagger) \text{ along } N(A), \tag{6-9}$$

$$AA^\dagger \text{ is a projection onto } R(A) \text{ along } N(A^\dagger)$$

where $R(\cdot)$ and $N(\cdot)$ denote the range and null spaces, respectively, of their arguments. That those projections are orthogonal follows from the relations [6-40]

$$R(A^\dagger) = N(A)^\perp, \quad N(A^\dagger) = R(A)^\perp$$

which can be deduced directly from Eq. (6-5) through (6-8); the superscript $\perp$ denotes the subspace orthogonal-complement relative to the usual inner product in finite-dimensional spaces. Complementary (orthogonal) projections are $I_\nu - A^\dagger A$ and $I_\mu - AA^\dagger$, respectively, where I_ν denotes the $\nu \times \nu$ identity matrix. The appropriate restriction of the domain of A is now evident; restricting A to $N(A)^\perp$ converts $A^\dagger A$ to the identity map on $N(A)^\perp$.

The Moore-Penrose inverse can always be represented explicitly. The decomposition of a general matrix in the form

$$A = BC$$

where $B: \mu \times \rho$, $C: \rho \times \nu$, and $\rho = \text{rank}(A)$, is always possible [6-41]. Representations for $B^\dagger$ and $C^\dagger$ are [6-42]

$$B^\dagger = (B^TB)^{-1}B^T, \quad C^\dagger = C^T(CC^T)^{-1}$$

and finally

$$A^{\dagger} = C^{\dagger}B^{\dagger}$$

However, it should be noted that the relation

$$(BC)^{\dagger} = C^{\dagger}B^{\dagger}$$

is not always true for arbitrary matrices B and C [6-43].

One of the chief applications of the Moore-Penrose inverse is in representing the solutions to linear algebraic equations of the (vector) form

$$Ax = b \tag{6-10}$$

If Eq. (6-10) is algebraically consistent, then its general solution can be written as [6-38]

$$x = A^{\dagger}b + (I_{\ell} - A^{\dagger}A)\omega \tag{6-11}$$

where ω is an arbitrary vector. Note that, since b is in R(A), Eq. (6-11) is an orthogonal decomposition (cf. Eq. (6-9)). If Eq. (6-10) is not algebraically consistent, then the product $A^{\dagger}b$ is the least-squares approximation of minimum norm. Further details on the Moore-Penrose and other generalized inverses may be found in [6-44].

6.4.2 Singular Value Decomposition

The symmetric products A^TA and AA^T of a rectangular matrix A have the same nonzero eigenvalues, and each has the same rank as that of A. The nonnegative square roots of the (real) eigenvalues of A^TA are called singular values of A. They have importance for numerical linear algebra because they are much less sensitive to perturbation in the elements of A than are the eigenvalues of A [6-23].

Any rectangular matrix A: $\mu \times \nu$ with rank ρ can be decomposed to display its singular values $\sigma_1 \geq \ldots \geq \sigma_\rho > 0 = \sigma_{\rho+1} = \ldots = \sigma_n$ as follows [6-34]

$$A = U\Sigma V^T \tag{6-12}$$

where U: $\mu \times \mu$ and V: $\nu \times \nu$ are orthogonal (unitary if A is complex) matrices, and Σ: $\mu \times \nu$ is a partitioned matrix with $D = \text{diag}(\sigma_1,\ldots,\sigma_\rho)$ in the upper left $\rho \times \rho$ block; the remaining blocks are zero. The simplicity of existence proofs for this decomposition [6-34, 6-45] stands in sharp contrast to the involved details of reliable algorithms for computing it [6-33]. This is a reflection of the comparative difficulty between analytical and computational determination of matrix rank since the number of nonzero singular values is equal to the rank of the matrix.

The columns of U and V in the singular value decomposition Eq. (6-12) have an important geometrical interpretation. If $U \equiv [U_1 \vdots U_2]$ and $V \equiv [V_1 \vdots V_2]$ are partitioned to be compatible with Σ, it follows from writing Eq. (6-12) in the form $AV = U\Sigma$ that [6-33]

$$\text{Columns of } U_1 (U_2) \text{ form a basis for } R(A)\ (R(A)^{\perp})$$

$$\text{Columns of } V_1 (V_2) \text{ form a basis for } N(A)^{\perp}\ (N(A)) \tag{6-13}$$

These interpretations are used to develop an expression for the general solution of the Kosut gain equation, Eq. (6-3), appropriate for the intended application.

Finally, it should be observed that the Moore-Penrose inverse is easily expressed in terms of this decomposition

$$A^{\dagger} = V\Sigma^{\dagger}U^{T} \tag{6-14}$$

where the only nonzero block of $\Sigma^{\dagger}$ is the upper left $\rho \times \rho$ block $D^{-1} = \text{diag}(\sigma_1^{-1},\ldots,\sigma_\rho^{-1})$.

6.4.3 An Expression for the General Solution

In examining solutions of the matrix Eq. (6-3), it is sufficient to consider vector equations of the form of Eq. (6-10). It is assumed that Eq. (6-10) is algebraically consistent. It was shown previously [6-3] that the general solution to Eq. (6-10) can be expressed in the form

$$x = A^{\dagger}b + S\gamma \tag{6-15}$$

where the columns of $S: \ell \times (\ell - r)$ form a basis for $N(A)$ and $\gamma: (\ell - r) \times 1$ is arbitrary. The classical expression Eq. (6-11) for the general solution of Eq. (6-10) differs from Eq. (6-15) in a qualitative manner that is important for the intended application; namely, it appears that there are more arbitrary parameters available. This is misleading.

Theorem 6-1

Equations (6-11) and (6-15) are equivalent solutions of Eq. (6-10).

Proof

If A has maximum rank, there is nothing to prove, since then $A^\dagger A = I_\ell$ and S is the empty matrix. Assume rank $(A) = r < \ell$.

($\rightarrow$) Let $\omega: \ell \times 1$ be arbitrary, and investigate the solvability of the linear system

$$S\gamma = (I_\ell - A^\dagger A)\omega$$

for $\gamma: (\ell - r) \times 1$. It follows from Eq. (6-9) that $I_\ell - A^\dagger A$ is a projection onto $N(A)$. Since the columns $s^1,\ldots,s^{\ell-r}$ of S form a basis for $N(A)$, there exist numbers $\gamma \equiv (\gamma_1,\ldots,\gamma_{\ell-r})$ such that

$$S\gamma \equiv \sum_{i=1}^{\ell-r} \gamma_i s^i = (I_\ell - A^\dagger A)\omega$$

($\leftarrow$) Let $\gamma: (\ell - r) \times 1$ be arbitrary, and investigate the solvability of the linear system

$$(I_\ell - A^\dagger A)\omega = S\gamma$$

for ω. By the definition of S: $S\gamma \in N(A)$. Since $I_\ell - A^\dagger A$ is a projection with range equal to $N(A)$, there exists such an ω. ∎

It is clear from this result that there are at most $\ell - r$ "effectively independent" choices of free parameters in the general solution of Eq. (6-10). The representation in Eq. (6-15) is therefore more appropriate for the application intended, and it is the foundation for the solution procedure to be developed. More precisely, the matrix form of Eq. (6-15) corresponding to Eq. (6-3) is used

$$X = A^{\dagger}B + S\Gamma$$

where Γ: $(\ell - r) \times m$ is arbitrary. Note that the matrix V_2 of Eq. (6-13) forms a natural choice for a basis of N(A) and is available directly from the decomposition, Eq. (6-12).

6.4.4 A Numerical Procedure

In the numerical procedure to be described, the focus is on the case that the coefficient matrix A in Eq. (6-3) is rank deficient. Otherwise, the solution is unique, there are no free parameters, and standard linear-equation-solving techniques can be used unless there is some reason to suspect A to be ill-conditioned with respect to matrix inversion. The rank-deficient case is the most interesting because of its beneficial potential for the underlying controller design problem.

The procedure consists of three basic parts: determining the numerical rank of A; computing the particular solution $A^{\dagger}B$; and computing the general solution $A^{\dagger}B + S\Gamma$.

6.4.4.1 The Procedure

Step 1. Determine the numerical rank of the coefficient matrix
(Denote by $rk_n(\cdot)$ the numerical rank of the indicated matrix.)

(a) Form the product $A \triangleq C_c P C_c^T$ (cf. Eq. (6-4)).

(b) Compute the singular value decomposition of A, obtaining matrices U and V, and singular values $\sigma_1, \ldots, \sigma_\ell$ (cf. Eq. (6-12)).

(c) Make an intelligent judgment as to which singular values $\sigma_{r+1}, \ldots \sigma_\ell$ should be considered zero (i.e., those considered to consist simply of an accumulation of roundoff errors) (cf. Note 1).

(d) Tentatively assign $rk_n(A) = r$.

(e) Estimate the numerical rank of the sensor matrix C_c by repeating steps corresponding to Items (b) and (c). If $rk_n(C_c) \neq rk_n(A)$, stop execution and investigate (cf. Note 2).

(f) Form the product $B \triangleq C_c P F^{*T}$ (cf. Eq. (6-4)).

(g) Estimate the numerical rank of the augmented matrix $[A \vdots B]$ by repeating steps corresponding to Items (b) and (c). If $rk_n([A \vdots B]) \neq rk_n(A)$, stop execution and investigate (cf. Note 3).

(h) When the estimates for numerical rank in Steps (e), (f), and (g) agree, assign to $rk_n(A)$ their common value.

Step 2. Determine the particular solution $A^{\dagger}B$.

(a) Compute $A^{\dagger}$ using the matrices U and V from Step 1(a) and the diagonal matrix $D_0^{-1} = \text{diag}\,(\sigma_1^{-1},\ldots,\sigma_{r_0}^{-1})$, where $r_0 \triangleq rk_n(A)$ (cf. Eq. (6-14)).

(b) Compute $A^{\dagger}B$ (cf. Note 4).

Step 3. Determine the general solution $A^{\dagger}B + S\Gamma$

(a) Select a matrix Γ: $(\ell - r_0) \times m$ of parameter values (arbitrary).

(b) Choose a matrix S: $\ell \times (\ell - r_0)$ with columns that form a basis for N(A) (cf. Note 5).

(c) Compute the general solution $X = A^{\dagger}B + S\Gamma$ (cf. Note 6).

6.4.4.2 Notes Referenced in the Procedure

(1) This task is nontrivial, and an understanding of the underlying physical problem is usually essential [6-46]. The purpose of Steps 1(e) and 1(g) is to contribute some of this insight to the final assignment of a value to $rk_n(A)$.

(2) Theoretically, rank (C_C) = rank (A) [6-3]. In most cases, computation of the numerical rank of C_C will be subject to less error than computation of the numerical rank of A. In some cases, it may be possible to determine rank (C_C) by inspection, which then determines the numerical rank of A.

(3) Theoretically, rank $([A \vdots B])$ = rank (A) [6-3].

(4) In actual computation, $A^{\dagger}$ need not be calculated explicitly; the solution $A^{\dagger}B$ is more efficiently computed as the product (cf. Eq. (6-14))

$$A^{\dagger}B = (V\Sigma^{\dagger})(U^{T}B)$$

(5) The basis chosen need not be a mutually orthogonal one. However, an orthogonal basis is readily available as the set of those columns of V (cf. Eq. (6-12)) which correspond to the zero singular values of A (cf. Eq. (6-13)).

(6) For a given control design problem, Steps 1 and 2 of the procedure need only be done once. Step 3 may be computed repeatedly with different choices for Γ and S until satisfactory control system performance improvement is achieved.

6.4.5 Summary

In order to be able to take advantage of the design freedom inherent in the extensions [6-3] to the Kosut method of suboptimal output feedback, a numerical procedure has been developed to compute the general solution to the gain equation in the rank-deficient case. The procedure is based upon a computation of the singular value decomposition for the coefficient matrix and expresses the solution in a form that facilitates exploiting the available design freedom. The design freedom consists of:

(1) A choice of the basis for the null space of the coefficient matrix.

(2) A choice of free parameters proportional in number to the rank deficiency of the coefficient matrix.

One candidate for a choice of the basis vectors is provided explicitly by the procedure. Choice of the free parameters may be motivated by consideration of the full-order closed-loop system equations that contain the reduced-order controller under study.

6.5 Conclusion

Considerable effort has been devoted to characterizing the design freedom with suboptimal output feedback that was recently uncovered and to developing methods for exploiting that freedom. In the present section, this effort has been placed in a much broader context by showing the close relationship with other significant work relating to design freedom and indicating the rapid development of interest in such matters. We believe that, in response to increasingly stringent spacecraft performance specifications, an increasing amount of attention will be given to discovering, characterizing, and developing systematic approaches to exploiting available freedom of choice in design.

6.6 Addenda

6.6.1 Characterization of Design Freedom with State Feedback ($m > 1$)

The system of Eq. (6-1) is being considered. In order to facilitate a concise statement of the main result, several preliminary ideas are necessary. First, recall an alternate characterization of controllability in the frequency domain:

Fact 6-1 [6-47; pp. 71, 161]. The pair (A,B) is controllable if and only if the polynomial matrix $\lambda \to S(\lambda) \triangleq [\lambda I_n - A \vdots B]$ has linearly independent rows for each complex number λ. ■

Second, note a particular property of such polynomial matrices:

Fact 6-2 [6-47; p. 194]. If (A,B) is controllable, there exists a matrix function $\lambda \to \begin{bmatrix} N(\lambda) \\ \hline M(\lambda) \end{bmatrix}$ which, for each complex number λ, forms a basis for the null space of $S(\lambda)$. ■

The main result is stated in terms of the $n \times m$ matrix function $N(\cdot)$:

Theorem 6-2 [6-13]

Assume that the pair (A,B) is controllable and that a set of distinct complex numbers λ_i, $i = 1,\ldots,n$ (containing the conjugates of each of its elements) is given. Then there exists a real full-state feedback matrix K (cf. Eq. (6-1)) such that

$$(A + BK)v^i = \lambda_i v^i \; , \; i = 1,\ldots,n$$

if and only if the (eigenvectors) v^i satisfy the following:

(1) The collection $\{v^i\}_{i=1}^n$ is linearly independent over the complex field.

(2) The collection of related pairs $\{(\lambda_i, v^i)\}_{i=1}^n$ are such that $v^j = \overline{v^i}$ if $\lambda_j = \overline{\lambda_i}$ (where "—" denotes complex conjugation)

(3) For each i: $v^i \in \text{span}\,[N(\lambda_i)]$.

Moreoever, any [illegible] K is unique. ■

This result can be modified appropriately to eliminate the need for assuming controllability of the pair (A,B) [6-11] or for assuming that the assigned closed-loop eigenvalues are distinct [6-12].

6.6.2 Exploitation of Design Freedom with State Feedback (m > 1)

A spectral characterization of the closed-loop output dynamics of Eq. (6-1) and (6-2) is given readily. For simplicity, assume distinct and stable closed-loop eigenvalues and a step input.

Fact 6-3 [6-13]. Assume the eigenvalues of A + BK are distinct with negative real parts. Let $\eta(t) \equiv \eta_s$ = constant. Then

$$y(t) - y_\infty = CVe^{\Lambda t}V^{-1}x(0) + CVe^{\Lambda t}\Lambda^{-1}V^{-1}BL\eta_s \ , \quad t \geq 0$$

and

$$y_\infty = -CV\Lambda^{-1}V^{-1}BL\eta_s$$

where $\Lambda \triangleq \mathrm{diag}\,(\lambda_i)$ displays the eigenvalues, and $V \triangleq [v^1 \vdots, \ldots, \vdots\, v^N]$ displays the eigenvectors, respectively, of A + BK. ■

The reader is referred to [6-13] for specific suggestions on how to use this representation effectively as a guide in selecting the eigenvectors to shape the dynamics of the response $y(t) - y_\infty$. In particular, the potential for eliminating effects of a given mode upon as many as m - 1 of the output components is of interest. It should be noted that much more freedom of choice really exists than is actually exploited in [6-13], since all but one of the eigenvectors are assumed fixed.

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SECTION 7

STOCHASTIC OUTPUT FEEDBACK COMPENSATORS
FOR DISTRIBUTED PARAMETER STRUCTURAL MODELS

7.1 Introduction

This section presents recent progress on the stochastic output feedback design problem for distributed parameter plants. The results presented are an extension of those described in Section 2 of Reference 7-1. The desire to design optimal output feedback compensators for distributed parameter flexible models motivated consideration of the following problem.

Given a cost functional $J(\cdot)$ defined over some set Γ of admissible compensator designs for the unknown distributed parameter plant P, determine the optimal design K^* that minimizes $J(\cdot)$ over Γ.

To accomplish this, define a sequence of finite-dimensional approximating plant models $\{P_n\}$ along with an associated sequence of cost functionals $\{J_n(\cdot)\}$ and determine the optimal compensator designs K_n^* that minimize $J_n(\cdot)$ over Γ. For "reasonable" choices of P, Γ, $J(\cdot)$, and the approximating scheme defining P_n and $J_n(\cdot)$, one would expect that the optimal compensators K_n^* should converge to K^* as $n \to \infty$. However, it is not obvious exactly what constitutes a "reasonable choice" of P, Γ, $J(\cdot)$, $\{P_n\}$, and $\{J_n(\cdot)\}$. For example [7-2], in the special case when P is an undamped flexible structure, linear regulators designed from truncated normal mode models P_n do not converge to a limiting linear regulator for the plant P. Consequently, it is extremely important to address the following fundamental questions:

(1) When does a set $S^* \subset \Gamma$ of optimal designs for the plant P exist?

(2) When do the corresponding sets S_n^* exist for the plants P_n?

(3) If the sets S_n^* exist for all n, does the sequence $\{S_n^*\}$ converge to some set $S \subset \Gamma$?

(4) If the sequence $\{S_n^*\}$ converges, does it converge to S^*?

Sufficient conditions are developed in this section for the answers to each of these questions to be at least approximately "yes". The results reported previously [7-1] addressed only the last of these questions, providing sufficient conditions for the optimality of the limiting set of compensators to which the sequence $\{S_n^*\}$ converged, <u>assuming the sequence converged</u>.

Continued investigation has shown that some of the key theorems employed in establishing these results are special cases of more general results [7-3]. In particular, the results of Reference 7-3 are sufficient to guarantee the existence of the sets S^* and S_n^*, the convergence of the sequence $\{S_n^*\}$, and the optimality of the limit. Further, these results are applicable to infinite-dimensional design sets Γ, which suggests the possibility of developing convergence conditions for infinite-dimensional compensators like the linear regulator considered in Reference 7-2. The price paid for this increased generality, however, is the necessity of dealing with weak convergence and related Banach space concepts. If these results are specialized to the original problem for which Γ is finite dimensional, the strong convergence results conjectured in the conclusions of Reference 7-1, Section 2 are obtained.

Consequently, the results of this section are presented in the following format. First, the notion of weak convergence and weak compactness are introduced in Section 7.2, and their importance is established in the context of the problem considered herein. Next, the general results of Reference 7-3 are presented in Section 7.3. Specifically, sufficient conditions on $J(\cdot)$, $J_n(\cdot)$, and Γ are presented for sets S^* and S_n^* of global minimizers of $J(\cdot)$ and $J_n(\cdot)$ to exist along with sufficient conditions to guarantee that the sequence $\{S_n^*\}$ converges weakly to a subset of S^* as $n \to \infty$. Once these general results are established, they are specialized in Section 7.4 to the original problem where the set Γ over which $J(\cdot)$ is minimized is finite dimensional. This yields a variety of existence and convergence results for the optimal fixed-form compensator design problem, which culminates in Theorem 7-6. Theorem 7-6 provides sufficient conditions for the answer to the four questions posed previously to be approximately "yes". The implications of the requirements and conclusions of Theorem 7-6 are discussed briefly in Section 7-5 to make these answers more precise.

7.2 Notion of Weak Convergence and Weak Compactness

In the results developed in Reference 7-1, the compactness of closed, bounded subjects of the set Γ of admissible compensators played an essential role. If Γ is defined on an infinite-dimensional Banach space, however, it is no longer necessarily true that closed, bounded subsets of Γ are compact. For example, the set $\{x \in B \mid \|x\| \leq M\}$ is compact if and only if the space B is finite dimensional [7-4]. Consequently, to extend the results of Reference 7-1 to problems for which Γ is defined on an infinite-dimensional Banach space, it is necessary to introduce the notions of weak convergence and weak compactness.

First, recall that a Banach space is a normed linear vector space that is complete in its norm (e.g., given a sequence $\{x_n\}$ such that $x_n \in B$ for all n and $\|x_n - x\| \to 0$ as $n \to \infty$, it follows that $x \in B$). Convergence of this type (e.g., given $\varepsilon > 0$, there exists N such that $n \geq N \Rightarrow ||x_n - x|| < \varepsilon$ for all $n \geq N$) is a strong convergence, denoted $x_n \overset{s}{\rightarrow} x$, and it corresponds to the usual notion of convergence in the Euclidean space R^n for any finite n.

If B is an infinite dimensional Banach space, then a weaker type of convergence is possible. Defining B^* as the dual space of B (i.e., the space of bounded linear functionals mapping B into R^1), {x} converges weakly to x ($x \overset{w}{\rightarrow} x$) if $|f(x_n) - f(x)| \rightarrow 0$ as $n \rightarrow \infty$ for all $f \in B^*$. The implication that convergence implies weak convergence is a standard result [7-4, 7-5], but the converse is not true unless B is finite dimensional.

Since B^* is a Banach space, the second dual space B^{**} of B is well defined and it is another standard result [7-6] that B may be identified with a subspace $\hat{B}$ of B^{**}. In general, this inclusion is proper, but a reflexive Banach space is one for which $\hat{B} = B^{**}$. This is a very broad class of spaces, which includes all Hilbert spaces and, as a consequence, all finite-dimensional Euclidean spaces R^n. For the problem considered here, the most significant feature of this class of spaces is that all bounded sequences $\{x_n\}$ (i.e., all sequences such that $||x_n|| < M$ for all n) have weakly convergent subsequences [7-5]. Consequently, any countable, bounded sequence on a reflexive Banach space may be decomposed into (possibly a countably infinite number of) convergent subsequences, each of which converges weakly to some limit in B.

This fact may be used to define the notion of weak compactness on reflexive Banach spaces. Specifically, in analogy with sequential compactness, a set S is weakly sequentially compact if every sequence $\{x_n\}$ in S has a weakly convergent subsequence that converges to an element of S. Since compactness and sequential compactness are equivalent on metric spaces [7-4], the term "weak sequential compactness" can be shortened to "weak compactness" or "w-compactness". Similarly, weakly closed (w-closed) sets may be defined as sets containing all of their weak limit points (i.e., S is w-closed if $x_n \in S$ for all n and $x_n \overset{w}{\rightarrow} x$ together imply $x \in S$). It then follows from the preceding result that if S is a weakly closed, bounded subset of a reflexive Banach space B, then S is weakly compact. Thus, the concepts of weak convergence and weak compactness play the same roles in infinite-dimensional, reflexive Banach spaces that their strong counterparts play in finite-dimensional spaces.

Finally, it is important to note that weak closedness is a "stronger" notion than ordinary (i.e., strong) closedness in spite of its name. Specifically, if a set is weakly closed, it is also strongly closed since if $x_n \in S$ for all n and $x_n \overset{s}{\rightarrow} x$ then $x_n \overset{w}{\rightarrow} x$, implying $x \in S$. The converse, however, is not generally true. A special case for which the two concepts are equivalent occurs, though, when S is convex also [7-3] (i.e., given $x_1, x_2 \in S$, it follows that $\lambda x_1 + (1 - \lambda)x_2 \in S$ for all $0 \leq \lambda \leq 1$). Thus, if a subset S of a reflexive Banach space B is closed, bounded, and convex, it is weakly compact.

7.3 General Existence and Convergence Results

The following paragraphs summarize the pertinent results of Reference 7-3 concerning the existence of global minimizers x^* of a functional $f(\cdot)$ defined on a reflexive Banach space B and the weak convergence of the global minimizers x_n^* of a sequence of approximating functionals $\{f_n(\cdot)\}$ to x^* as $n \to \infty$. In particular, very general sufficient conditions are developed for both of these situations to occur. More specifically, the first problem considered is as follows.

Given $f(\cdot): E \to R^1$ for some $E \subset B$, what conditions on E and $f(\cdot)$ are sufficient to guarantee the existence of a set $S^* \subset E$ such that $x^* \in S^*$ implies $f(x^*) \leq f(x)$ for all $x \in E$? First, define the following constructs. The functional $f(\cdot)$ is lower semicontinuous (lsc) on the set E if [7-3, 7-7] the set $\{x \in E | f(x) > \lambda\}$ is open for any real λ or the set $\{x \in E | f(x) \leq \lambda\}$ is closed for any real λ. Alternatively, define lsc functions sequentially [7-5, 7-7, 7-8] (i.e., $f(\cdot)$ is lsc on E if for any sequence $\{x_n\} \in E$ with $x_n \xrightarrow{s} x$, it follows that $f(x) \leq \liminf_{n\to\infty} f(x_n)$). Strictly speaking, this latter description defines a sequentially lower semicontinuous function, but it is noted in Reference 7-3 that while considering strong convergence, the two concepts are equivalent [c.f. 7-7]. This is not the case, however, if weak convergence is considered. That is, we can define $f(\cdot)$ as a weakly lower semicontinuous (wlsc) function on E if for any real λ, the set $\{x \in E | f(x) \leq \lambda\}$ is weakly closed, and define $f(\cdot)$ as a weakly sequentially lower semicontinuous (wslsc) function on E if for any $\{x_n\} \in E$ with $x_n \xrightarrow{w} x$ it follows that $f(x) \leq \liminf_{n\to\infty} f(x_n)$. It is noted in Reference 7-3 that weak sequential lower semicontinuity is generally a weaker property than weak lower semicontinuity, i.e., if $f(\cdot)$ is wlsc, it is wslsc, but the converse is not generally true.

Similarly, as in the case of strongly and weakly closed sets, it is important to note that weak lower semicontinuity is a "stronger" concept than (strong) lower semicontinuity. Specifically, if $f(\cdot)$ is wlsc on E, then the set $\{x \in E | f(x) \leq \lambda\}$ is weakly closed, hence closed for all real λ, which implies that $f(\cdot)$ is lsc on E. Again, as in the case of strongly and weakly closed sets, the converse does hold if additional conditions are imposed on $f(\cdot)$. Specifically, the functional $f(\cdot)$ is quasiconvex [7-3] on the convex set C if the set $\{x \in C | f(x) \leq \lambda\}$ is convex for all real λ. Thus, if E is convex and weakly closed and $f(\cdot)$ is lsc and quasiconvex on E, the set $\{x \in E | f(x) \leq \lambda\}$ is (strongly) closed and convex, hence weakly closed, implying $f(\cdot)$ is wlsc on E. Note that the notion of a quasiconvex functional is a generalization of the more common notion of a convex functional, which is defined on a convex set C as one for which

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for any x_1, $x_2 \in C$ and $0 \leq \lambda \leq 1$. It is noted in Reference 7-3 that a quasiconvex functional is one for which

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \max[f(x_1), f(x_2)]$$

for any x_1, $x_2 \in C$ and $0 \leq \lambda \leq 1$. Since

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \leq \max[f(x_1), f(x_2)]$$

for all $0 \leq \lambda \leq 1$, it is clear that $f(\cdot)$ is quasiconvex if it is convex.

Given the definition of weak sequential lower semicontinuity just presented and the notions of weak convergence and weak compactness presented in Section 7.2, it is simple to establish the following existence result for S^*.

Theorem 7-1 ([7-3], Theorem 1.4.1)

If $f(\cdot): E \to R^1$ is wslsc on the weakly compact set E, there exists a nonempty set $S^* \subset E$ such that $x^* \in S^*$ implies $f(x^*) = \inf_{x \in E} f(x)$.

Proof

Let

$$f^* = \inf_{x \in E} f(x)$$

By definition of inf f(x), there exists a sequence $\{x_n\} \subset E$ such that

$$\lim_{n \to \infty} f(x_n) = f^*$$

Since E is w-compact, there exists some x^* ε E and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{w} x^*$. Clearly, since $f(x_n) \to f^*$, $f(x_{n_k}) \to f^*$ also. Thus, since $f(\cdot)$ is wslsc

$$x_{n_k} \xrightarrow{w} x^* \Rightarrow f(x^*) \leq \liminf_{n\to\infty} f(x_{n_k}) = f^* \leq f(x^*)$$

Thus, $x^* \varepsilon S^*$. ■

Unfortunately, in the applications considered here, the set E (corresponding to the set Γ of admissible compensators in the optimal compensator design problem) generally is not bounded, hence not w-compact. If additional constraints are imposed on $f(\cdot)$, however, it is possible to guarantee that $f(\cdot)$ exhibits a set of global minima on some w-compact subset E' of E. Specifically, $f(\cdot)$ satisfies a T-property on E if there exist x_0 ε E and $T_0 > 0$ such that $||x - x_0|| \geq T_0$ implies $f(x) > f(x_0)$ for any x ε E. As the following theorem shows, if $f(\cdot)$ satisfies a T-property on a w-closed set E, the optimization can be restricted to some w-compact subset of E.

Theorem 7-2 (adapted from [7-3], Theorem 1.4.2)

Suppose E is a weakly closed subset of a reflexive Banach space B. If $f(\cdot): E \to R^1$ is wslsc and satisfies a T-property on E, then there exists a w-compact subset C of E and a set $S^* \subset C$ such that $x^* \varepsilon S^*$ implies

$$f(x^*) = \inf_{x \varepsilon E} f(x)$$

Proof

Let $S(x_0,T_0)$ be a strongly closed sphere of radius T_0 about x_0 ε E. Since the set is strongly closed and convex, it is weakly closed and bounded, hence weakly compact.

Since E is weakly closed, the set

$$C = \{x \varepsilon E \mid ||x - x_0|| \leq T_0\} = S(x_0,T_0) \cap E$$

is also weakly compact.

Since $f(\cdot)$ satisfies the T-property, $f(x) > f(x_0)$ for any $x \in E$ but $x \notin C$. Thus

$$\inf_{x \in E} f(x) = \min \left[\inf_{x \in C} f(x), \inf_{\substack{x \notin C \\ x \in E}} f(x) \right] = \inf_{x \in C} f(x)$$

Since C is w-compact, the existence of a set S^* of global minimizers of $f(\cdot)$ over C is guaranteed by Theorem 7-1. ∎

It is possible to establish that $f(\cdot)$ satisfies a T-property in a variety of ways. One of the simplest is to guarantee that $f(x)$ increases as $||x||$ increases. For example, $f(\cdot)$ is semicoercive on E if [7-7]

$$\lim_{\substack{||x|| \to \infty \\ x \in E}} \frac{f(x)}{||x||} > 0$$

and coercive if

$$\lim_{\substack{||x|| \to \infty \\ x \in E}} \frac{f(x)}{||x||} = +\infty$$

from which it is immediately clear that any coercive function is semicoercive.

<u>Lemma 7-1</u>

If $f(\cdot): E \to R^1$ is semicoercive on E, it satisfies a T-property.

<u>Proof</u>

Since $f(\cdot)$ is semicoercive, there exists some $\alpha > 0$ such that

$$\lim_{\substack{||x|| \to \infty \\ x \in E}} \frac{f(x)}{||x||} \geq \alpha$$

Thus, there is some constant $M > 0$ such that

$$||x|| \geq M,\ x \in E \Rightarrow \frac{f(x)}{||x||} \geq \frac{\alpha}{2} \Rightarrow f(x) \geq \frac{\alpha}{2} ||x||$$

Now, choose any $x_0 \in E$ and define

$$M' = \max\left[M, \frac{2f(x_0)}{\alpha} + 1\right]$$

so that

$$\begin{aligned} ||x|| \geq M',\ x \in E \Rightarrow f(x) &\geq \frac{\alpha}{2}||x|| \\ &\geq \frac{\alpha}{2} M' \\ &\geq f(x_0) + \frac{\alpha}{2} \\ &> f(x_0) \end{aligned}$$

Finally, note that, by the triangle inequality

$$\begin{aligned} ||x - x_0|| \leq ||x|| + ||-x_0|| &= ||x|| + ||x_0|| \\ \Rightarrow ||x|| &\geq ||x - x_0|| - ||x_0|| \end{aligned}$$

Thus, if

$$T_0 = ||x_0|| + M'$$

it follows that

$$||x - x_0|| \geq T_0 \Rightarrow ||x|| \geq M' \Rightarrow f(x) > f(x_0)$$ ■

Once it has been established that some set S^* of global minimizers of $f(\cdot)$ over the weakly closed set E exist on some weakly compact set C, we can turn to the problem of how elements of the set S^* are to be constructed. Specifically, consider the sequence $\{f_n(\cdot)\}$ of approximating functionals defined on the approximating sets $\{E_n\}$ for $n = 1, \ldots$ The basic question considered here is "what relationship must exist between the approximating sequences $\{f_n(\cdot)\}$ and $\{E_n^*\}$ and $f(\cdot)$ and E to guarantee that solutions x_n^* of the approximating problems defined by $f_n(\cdot)$ and E_n converge in some useful sense to an element of S^*?" A very useful answer to this question is developed in Reference 7-3 by introducing the concept of a "consistent discretization" of the original problem. The remainder of this section is therefore devoted to presenting a slight generalization of these results as the basis for the results that follow. (The generalization is this: Reference 7-3 assumes that the set E over which $f(\cdot)$ is defined and minimized is the entire reflexive Banach space B. In the problem considered here, E may be any weakly closed subset of B.)

A discretization of the minimization problem of $f(\cdot)$ over C may be defined [7-3] as a family of subsets $\{E_n\}$ of reflexive Banach spaces $\{B_n\}$, a family of functionals $\{f_n(\cdot)\}$ defined on E_n, a family of mappings $\{p_n\}$ of E_n into E, a family of mappings $\{r_n\}$ of E into E_n, and a family of subsets $\{C_n\}$ of E_n. The basic idea is that $\{f_n(\cdot)\}$, $\{E_n\}$, $\{B_n\}$, and $\{C_n\}$ represent approximations, in some sense, of the components $f(\cdot)$, E, B, and C of the original minimization problem, respectively. The mappings $\{p_n\}$ and $\{r_n\}$ thus represent projections of the approximating sets $\{E_n\}$ onto the set E and the set E the approximating sets $\{E_n\}$, respectively. These definitions are general enough to allow the approximating problems and the original problem to be defined on Banach spaces $\{B_n\}$ and B that are qualitatively very different. In particular, the approximating spaces $\{B_n\}$ may be finite dimensional.

Using these definitions, Reference 7-3 associates the following approximate problem with each discretization. Given a sequence $\{\xi_n\}$ of positive numbers converging to zero, consider the set $\tilde{S}_n$ of elements $\tilde{x}_n \in C_n$ such that

$$f_n(\tilde{x}_n) \leq \inf_{x_n \in C_n} f_n(x_n) + \xi_n$$

Note that if there exists some subset S_n^* of C_n such that

$$f_n(x_n^*) = \inf_{x_n \varepsilon C_n} f_n(x_n)$$

for all $x_n^* \varepsilon S_n^*$, then it follows that

$$f_n(x_n^*) = \inf_{x_n \varepsilon C_n} f_n(x_n) \leq \inf_{x_n \varepsilon C_n} f_n(x_n) + \xi_n$$

for any $\xi_n > 0$, implying $S_n^* \subset \tilde{S}_n$.

In order to develop sufficient conditions for the set of $\tilde{S}_n$ to converge in some sense to S^*, Reference 7-3 defines a consistent discretization as a discretization of the original optimization problem that satisfies the following constraints

(1) The set $\tilde{S}_n$ exists for all n.

(2) $\limsup_{n\to\infty} [f(p_n\tilde{x}_n) - f_n(\tilde{x}_n)] < 0$ if $\tilde{x}_n \varepsilon \tilde{S}_n$.

(3) The sets $C^n = p_n C_n \cup C$ are uniformly bounded.

(4) If $z_{n_i} \varepsilon C^{n_i}$ and $z_{n_i} \overset{w}{\to} z$, then $z \varepsilon C$.

(5) For some $x^* \varepsilon S^*$, $\limsup_{n\to\infty} f_n(r_n x^*) < f(x^*)$.

(6) For the same $x^* \varepsilon S^*$, $r_n x^* \varepsilon C_n$.

Note that if $p_n C_n \subset C$ for all n, Conditions (3) and (4) are satisfied automatically. These conditions are sufficient to establish the following result.

<u>Theorem 7-3</u> ([7-3], Theorem 2.2.1)

Suppose $\{f_n(\cdot), E_n, B_n, C_n, p_n, r_n\}$ define a consistent discretization, and let K be a subset of E such that $C^n \subset K$ for all n sufficiently large. If C is weakly compact and $x_n \varepsilon S_n$ for all n, then

$$\lim_{n\to\infty} f(p_n \tilde{x}_n) = \lim_{n\to\infty} f_n(\tilde{x}_n) = f(x^*)$$

where x^* satisfies Conditions (5) and (6). Further, the sequence $\{p_n \tilde{x}_n\}$ may be decomposed into subsequences, each of which converges weakly to an element of S^*, i.e., $p_n \tilde{S} \overset{w}{\to} S \subset S^*$.

Before proceeding with the proof of Theorem 7-3, the following lemma, required in the proof, is established.

Lemma 7-2

The sequence $\{\gamma_n\}$ converges to zero where

$$\gamma_n = f(x^*) - \inf_{x \in C^n} f(x)$$

and $x^* \in S^*$. ■

Proof of Lemma 7-2

(1) First, note that since $x^* \in C \subset C^n$, $\gamma_n > 0$ for all n.

(2) Next, for each n, select some $z_n \in C^n$ such that

$$f(z_n) \leq \inf_{x \in C^n} f(x) + \frac{1}{n} = f(x^*) + [\frac{1}{n} - \gamma_n]$$

(3) Now, choose N such that $n \geq N \Rightarrow C^n \subset K \subset B$ and note that since K is bounded, the sequence $\{z_n\}$ is bounded for all $n \geq N$. Since B is a reflexive Banach space, $\{z_n\}$ may be decomposed into a collection of subsequences $\{z_{n_k}\}$, each of which converges weakly to some $z \in B$.

(4) By Condition (4) of the consistency requirements

$$z_{n_k} \xrightarrow{w} z \implies z \in C$$

so that $f(x^*) \leq f(z)$.

(5) Note also that since $f(\cdot)$ is wslsc on K

$$f(z) < \liminf_{k\to\infty} f\left(z_{n_k}\right)$$

$$< f(x^*) + \liminf_{k\to\infty} \left[\frac{1}{n_k} - \gamma_{n_k}\right]$$

$$\implies \liminf_{k\to\infty} \left[\frac{1}{n_k} - \gamma_{n_k}\right] > 0$$

(6) Thus

$$\liminf_{k\to\infty} \left[\frac{1}{n_k} - \gamma_{n_k}\right] \geq \liminf_{k\to\infty} \frac{1}{n_k} - \limsup_{k\to\infty} \gamma_{n_k}$$

$$\implies \liminf \frac{1}{n_k} \geq \limsup_{k\to\infty} \gamma_{n_k}$$

$$\implies 0 \geq \limsup_{k\to\infty} \gamma_{n_k}$$

$$\geq \liminf_{k\to\infty} \gamma_{n_k} \geq 0$$

$$\implies \lim_{k\to\infty} \gamma_{n_k} = 0$$

(7) Since this proof may be repeated for all subsequences $\{z_{n_k}\}$ defined in Item 3, it follows that $\lim_{n\to\infty} \gamma_n = 0$.

Proof of Theorem 7-3

(1) Take N and $\{\gamma_n\}$ as in Lemma 7-2. Since $x^* \in S^*$, it follows

$$f(x^*) = \inf_{x \in C^n} f(x) + \gamma_n$$

$$\leq f(p_n \tilde{x}_n) + \gamma_n \quad \text{for } \tilde{x}_n \in \tilde{S}_n \subset C_n$$

$$= f_n(\tilde{x}_n) + \gamma_n + \eta_n$$

where $\eta_n = f(p_n \tilde{x}_n) - f_n(\tilde{x}_n)$

$$\leq \inf_{x_n \in C_n} f_n(x_n) + \xi_n + \gamma_n + \eta_n \quad \text{since } \tilde{x}_n \in \tilde{S}_n$$

$$\leq f_n(r_n x^*) + \xi_n + \gamma_n + \eta_n \quad \text{since } r_n x^* \in C_n \text{ by}$$

Condition (6) of the consistency definition.

(2) Define $\delta_n = f_n(r_n x^*) - f(x^*)$ so that

$$(1) \Rightarrow f(x^*) \leq f(x^*) + \delta_n + \xi_n + \gamma_n + \eta_n$$

$$\Rightarrow \delta_n + \xi_n + \gamma_n + \eta_n \geq 0 \text{ for all } n$$

$$\Rightarrow 0 \leq \liminf_{n \to \infty} (\delta_n + \xi_n + \gamma_n + \eta_n)$$

$$\leq \limsup_{n \to \infty} (\delta_n + \xi_n + \gamma_n + \eta_n)$$

$$\leq 0$$

Since $\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \gamma_n = 0$ and, by consistency Conditions (2) and (5), respectively, it follows that

$$\limsup_{n\to\infty} \eta_n \leq 0$$

and

$$\limsup_{n\to\infty} \delta_n \leq 0$$

(3) Thus

$$\lim_{n\to\infty} (\delta_n + \eta_n) = 0$$

Note that if $\liminf_{n\to\infty} \eta_n \leq \alpha$ for some $\alpha < 0$, there exists some subsequence $\{\eta_{n_i}\}$ so that

$$\eta_{n_i} \leq \alpha/2 < 0$$

for all i, implying there is some subsequence $\{\delta_{n_i}\}$ such that

$$\delta_{n_i} \geq -\alpha/4 > 0$$

However, this implies

$$\limsup_{n\to\infty} \delta_n \geq -\alpha/4 > 0$$

which contradicts Condition (5) of the consistency requirements.

Thus

$$\liminf_{n\to\infty} \eta_n \geq 0 \geq \limsup_{n\to\infty} \eta_n \Rightarrow \lim_{n\to\infty} \eta_n = 0$$

Similarly, it follows that

$$\lim_{n\to\infty} \delta_n = 0$$

(4) Thus, it follows from the inequalities in Item (1) that

$$f(x^*) = \lim_{n\to\infty} f(p_n \tilde{x}_n) = \lim_{n\to\infty} f_n(\tilde{x}_n)$$

(5) Finally, since $C^n \subset K$ for all $n \geq N$ with K bounded, the sequence $(p_n x_n^*)$ may be decomposed into subsequences that converge weakly to limit points in B.

Since $\tilde{x}_n \in C_n$, $p_n \tilde{x}_n \in C^n$ for all n so that by consistency Condition (4), if $p_{n_k} \tilde{x}_{n_k} \overset{W}{\to} z$, then $z \in C$.

Further, since $f(\cdot)$ is wslsc on K

$$f(z) \leq \liminf_{k\to\infty} f(p_{n_k} \tilde{x}_{n_k}) = f(x^*)$$

so $z \in S^*$.

Thus $p_n \tilde{S}_n = \{p_n \tilde{x}_n | \tilde{x}_n \in \tilde{S}_n\} \overset{W}{\to} S \subset S^*$.

7.4 Application to Fixed-Form Compensator Design Problem

The results just presented are very general; thus, they are applicable to a very broad class of problems including the fixed-form compensator design problem of interest herein. Because of their generality, the conditions these results impose on arbitrary optimization problems are somewhat hard to interpret. However, if they are specialized to the optimal fixed-form compensator design problem, these conditions simplify considerably. Consequently, the remainder of this section is concerned with the following special case.

The arbitrary functional $f(\cdot)$ will be replaced by the cost functional $J(\cdot)$ defined over some set Γ of admissible compensator designs. It is important to note that this is a departure from traditional optimal control problems in which $J(\cdot)$ is a functional on some function space U of control inputs u(t). While it is true that a priori restriction of the control law to some prespecified form may result in a performance degradation (i.e., the set of Γ of admissible compensators may not be large enough to generate the control input $u^*(t)$ that minimizes J(u) over U), it is also true that practical designs will be restricted to some set of admissible compensators anyway. Thus, the approach developed is to identify the set Γ a priori on the basis of various important practical considerations and to optimize performance within these restrictions. In particular, it is assumed that once these restrictions have been identified, a compensator form meeting these restrictions is selected that reduces the design problem to that of selecting values for some finite number ℓ of design parameters that minimize the cost $J(\cdot)$. Thus, the set Γ of admissible compensator designs becomes a subset of the parameter space R^{ℓ}.

The most significant simplification inherent in this formulation of the compensator design problem is that optimization is now carried out over the finite-dimensional Euclidean space R^{ℓ} rather than the infinite-dimensional space U. Consequently, since weak and strong convergence are equivalent in finite-dimensional Banach spaces, the results of the preceding section can be used to obtain sufficient conditions for the strong convergence of the approximate compensator designs to the desired optimal limit. That is, by formulating the design procedure described in the introduction as a discretization of the minimization of $J(\cdot)$ over Γ, sufficient conditions for the convergence of the optimal parameter vectors $\underline{k}_n^*$ for the approximating plants P_n to the optimal parameter vectors $\underline{k}^*$ for the infinite-dimensional plant P may be obtained.

In particular, the discretization defined by the design procedures is the following. First, all minimizations will be restricted to the closed set $\Gamma \subset R^{\ell}$ (i.e., $B_n = B = R^{\ell}$ and $E_n = E = \Gamma$ for all n in the notation of the previous subsection). Consequently, the mappings of $\{p_n\}$ and $\{r_n\}$ appearing in the definition of a discretization are all identity mappings in this case. The approximate optimization problems associated with this discretization will be the minimization of $J_n(\cdot)$ over Γ (i.e., the set S_n^* of solutions to the n^{th} approximating problem consists of $\underline{k}_n^* \in \Gamma$ such that $J_n(\underline{k}_n^*) = \inf_{\underline{k} \in \Gamma} J_n(\underline{k})$. In order for the design problem considered here to be well posed, it is, of course, necessary that the sets $\{S_n^*\}$ exist for all n. If $J_n(\cdot)$ and $J(\cdot)$ are required to satisfy the conditions of Theorem 7-3, for example, the existence of the sets $\{S_n\}$ and S^* is guaranteed as are the existence of the compact sets $\{C_n\}$ and C containing $\{S_n^*\}$ and S^*, respectively. Note, however, that without further restrictions on the functions $\{J_n(\cdot)\}$, the compact sets $\{C_n\}$ need not bear any special relation to the set C. In particular, it is entirely possible that the sets $\{C_n\}$ are not uniformly bounded and will therefore not converge to the set C in any sense as $n \to \infty$, which suggests that the sets $\{S_n^*\}$ may not converge into the set S^* as $n \to \infty$. If, however, it can be established that the discretization just defined satisfies the consistency requirements developed in the preceding subsection, it will follow from Theorem 7-3 that the sets S_n^* of optimal designs associated with the plants P_n will converge to to a subset of the set S^* of optimal designs associated with the plant P and that $J_n(\underline{k}_n^*) \to J(\underline{k}^*)$ for any $\underline{k}_n^* \in S_n^*$ and $\underline{k}^* \in S^*$. Specifically, combining the definition of a consistent discretization with the conditions of Theorem 7-3 and specializing them to the present problem, the following result is immediate.

<u>Theorem 7-4</u>

Suppose Γ is a closed set in R^ℓ and $J(\cdot): \Gamma \to R^1$ is an lsc function exhibiting some bounded set S^* of global minima over Γ. Suppose further that each function $J_n(\cdot): \Gamma \to R^1$ also exhibits some bounded set S_n of global minima over Γ and that the following conditions are met:

(1) $S_n^* \subset K$, $S^* \subset K$ for some compact set $K \subset \Gamma$.

(2) $\limsup_{n\to\infty} \left[J(\underline{k}_n^*) - J_n(\underline{k}_n)\right] \leq 0$ if $\underline{k}_n^* \in S_n^*$.

(3) $\limsup_{n\to\infty} J_n(\underline{k}^*) \leq J(\underline{k}^*)$ for some $k^* \in S^*$.

Under these conditions, the following results hold:

(4) $S_n^* \to S \subset S^*$.

(5) $\lim_{n\to\infty} J_n(\underline{k}_n^*) = \lim_{n\to\infty} J(\underline{k}_n^*) = J(\underline{k}^*)$

where $\underline{k}_n^* \in S_n^*$ and $\underline{k}^* \in S^*$ satisfies Condition (3).

Note here that the compact set K defined in Condition (1) takes the place of the sets $\{C_n\}$ and C in the original definition of a consistent discretization. This guarantees that Conditions (3), (4), and (5) of the original definition are satisfied. Given the assumed boundedness of sets S^* and S_n^* individually, it is clear that Condition (1) of Theorem 7-4 can be satified if and only if the sets $\{S_n^*\}$ are uniformly bounded.

If Condition (1) of Theorem 7-4 is satisfied, Conditions (2) and (3) may be replaced by a uniform convergence condition, as the following lemma demonstrates.

Lemma 7-3

If Condition (1) of Theorem 7-4 is satisfied and if $J_n(\cdot) \to J(\cdot)$ uniformly on the set K, then Conditions (2) and (3) are also satisfied.

Proof

(1) Condition (3) follows immediately since if $J_n(\cdot) \to J(\cdot)$ uniformly on K, $\underline{k}^* \varepsilon\ S^* \subset K \Rightarrow J_n(\underline{k}^*) \to J(\underline{k}^*)$.

Thus

$$\limsup_{n\to\infty} J_n(\underline{k}^*) = \lim_{n\to\infty} J_n(\underline{k}^*) = J(\underline{k}^*)$$

(2) Similarly, since $J_n(\cdot) \to J(\cdot)$ uniformly on K, $J(\underline{k}) - J_n(\underline{k}) \to 0$ uniformly for all $\underline{k}\ \varepsilon$ K.

In particular, since $\underline{k}_n^* \varepsilon\ S_n^* \subset K$ for all n, $J(\underline{k}_n) - J_n(\underline{k}_n) \to 0$ as $n \to \infty$.

Thus

$$\limsup_{n\to\infty} [J(\underline{k}_n) - J_n(\underline{k}_n)] = \lim_{n\to\infty} [J(\underline{k}_n) - J_n(k_n)] = 0$$

and Condition (2) is satisfied. ∎

To establish Condition (1), either one of two approaches may be taken, resulting in either a constrained optimization problem or an unconstrained optimization problem. In the first case, if $J(\cdot)$ is required to satisfy a T-property, then it follows from Theorem 7-2 that there exists some compact set C containing the set S^* of all global minimizers of $J(\cdot)$ over Γ. Consequently, if each function $J_n(\cdot)$ is lower semicontinuous on Γ, the existence of nonempty sets $S_n^{**} \subset C$ that minimize $J_n(\cdot)$ over C is guaranteed by Theorem 7-1. Thus, applying Theorem 7-4 to the sets S_n^{**} leads immediately to Theorem 7-5.

Theorem 7-5 (Constrained Optimization)

Suppose Γ is a closed set in R^ℓ and $J(\cdot): \Gamma \to R^1$ is an lsc function satisfying a T-property on Γ. This guarantees that $S^* \subset C$ for some compact set C. Suppose $\{J_n(\cdot)\}$ is a sequence of lsc functions on C that converges uniformly to $J(\cdot)$ on C, and let S_n^{**} denote the set of global minima of $J_n(\cdot)$ over C, guaranteed to exist by Theorem 7-1.

Under these conditions, the following results hold:

(1) $S_n^{**} \to \hat{S} \subset S^*$ as $n \to \infty$

(2) $\lim\limits_{n\to\infty} J_n(\underline{k}_n^{**}) = \lim\limits_{n\to\infty} J(\underline{k}_n^{**}) = J(\underline{k}^*)$

where $\underline{k}_n^{**} \in S_n^{**}$ and $\underline{k}^* \in S^*$.

It is extremely important to note here that the sets $\{S_n^{**}\}$ are not the the same as the sets $\{S_n^*\}$ considered earlier. For example, suppose $J(\cdot)$ satisfies the conditions of Theorem 7-5 and $\inf\limits_{\underline{k}\in\Gamma} J(\underline{k}) > 0$. Then if the approximating functions $\{J_n(\cdot)\}$ are defined by

$$J_n(k) = \begin{cases} J(k) & , \quad ||k|| < n \\ 0 & , \quad ||k|| \geq n \end{cases}$$

for all $k \in \Gamma$, it follows that for n such that $C_n \subset L_n = \{k \in \Gamma \mid \|k\| < n\}$ $S_n^{**} = S^*$, guaranteeing that $S_n^{**} \to S^*$ as $n \to \infty$ in agreement with Theorem 7-5. The sets S_n^*, however, consist of all elements $\hat{\underline{k}}$ of Γ not in L_n, ensuring that

$$J_n(\hat{\underline{k}}) = 0 = \inf_{k \in \Gamma} J_n(\underline{k})$$

Note that $J_n(\underline{k})$ is lsc on Γ since $J(\underline{k})$ is lsc on L_n, and 0 is obviously lsc on $\Gamma - L_n$. Thus, the only points in question occur on the boundary Ω between L_n and $\Gamma - L_n$. Since L_n is open, $\Gamma - L_n$ is closed so $\Omega \subset \Gamma - L_n$ implying

$$J(\underline{k}) = 0 \leq \liminf_{n \to \infty} J(\underline{k}_n)$$

if $\underline{k} \in \Omega$ for any sequence $\{\underline{k}_n\}$ in Γ. Note also for any compact set $K \subset \Gamma$, there is some N such that $J_n(\underline{k}) = J(\underline{k})$ for all $\underline{k} \in K$ if $n \geq N$. Thus, $\{J_n(\cdot)\}$ converges uniformly to $J(\cdot)$ on any compact subset of Γ.

To avoid these difficulties, the second approach to establishing that Condition (1) of Theorem 7-4 holds is to strengthen the conditions on $\{J_n(\cdot)\}$ and $J(\cdot)$ as follows. First, uniform convergence will be required on all of Γ rather than just compact subsets of Γ. This requirement will exclude the pathologies exhibited by the last example, although even under this restriction, another difficulty can arise. In particular, the fact that $J(\cdot)$ satisfies a T-property is not enough to guarantee that $J(\underline{k})$ does not approach $\inf_{\underline{k} \in \Gamma} J(\underline{k})$ as $\|\underline{k}\| \to \infty$. For example, consider the one parameter cost function illustrated in Figure 7-1. Here, $J(k)$ satisfies a T-property at $k = 0$, but $J(k) \to J(0)$ as $k \to \pm\infty$. To correct this difficulty, define a sharp T-property as follows.

Definition (Sharp T-Property)

The function $J(\cdot): \Gamma \to R^1$ satisfies a sharp T-property if there exist $\underline{k}_1, \underline{k}_2 \in \Gamma$ and $T_0 > 0$ such that

$$\|\underline{k} - \underline{k}_1\| \geq T_0 \Rightarrow J(\underline{k}) > J(\underline{k}_1) > J(\underline{k}_2)$$

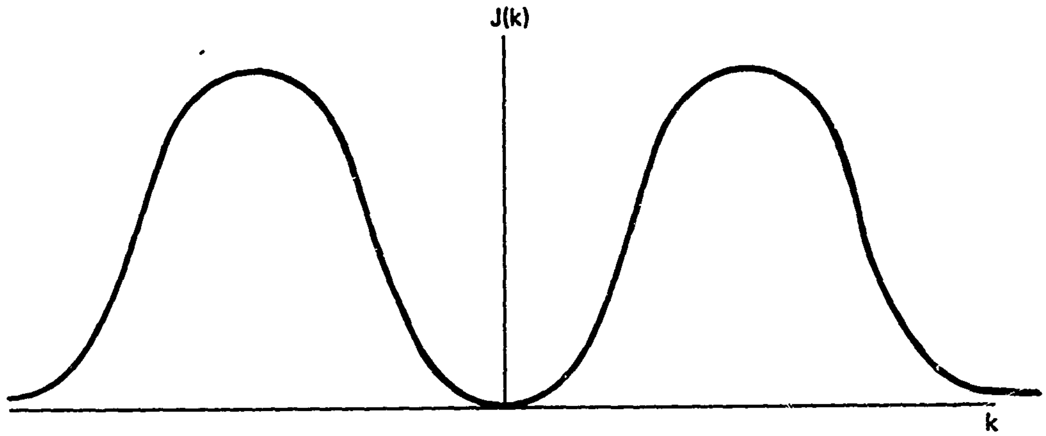

Figure 7-1. $J(k)$ satisfies a T-property at $k = 0$ but $J(k) \to \inf_{k \in R^1} J(k)$ as $k \to \pm\infty$

It is clear from this definition that any $J(\cdot)$ satisfying a sharp T-property satisfies a T-property and that if

$$L = \{k \in \Gamma \mid \|\underline{k} - \underline{k}_1\| \leq T_0\}$$

L is a compact set and

$$\inf_{\underline{k} \notin L} J(\underline{k}) \geq J(\underline{k}_1) > J(\underline{k}_2) \geq \inf_{k \in L} J(\underline{k})$$

It is also clear that if $J(\cdot)$ is a semicoercive function, it satisfies a sharp T-property. With these results, the following unconstrained optimization theorem may be established.

<u>Theorem 7-6 (Unconstrained Optimization)</u>

Suppose Γ is a closed set in R^ℓ and $J(\cdot): \Gamma \to R^1$ is an lsc function satisfying a sharp T-property on Γ. If $\{J_n(\cdot)\}$ is a family of lsc functions on Γ that converges uniformly to $J(\cdot)$ on Γ, then the following conditions hold:

(1) All global infima of $J_n(\cdot)$ and $J(\cdot)$ are achieved on some compact set K, i.e., $S_n^*, S^* \subset K$.

(2) $S_n^* \to \hat{S} \subset S^*$ as $n \to \infty$.

(3) $\lim_{n\to\infty} J_n(\underline{k}_n^*) = \lim_{n\to\infty} J(\underline{k}_n^*) = J(\underline{k}^*)$

where $\underline{k}_n^* \in S_n^*$ and $\underline{k}^* \in S^*$.

Proof

(1) Since $J(\cdot)$ satisfies a sharp T-property on Γ, there exists some compact set K such that

$$\inf_{\underline{k}\notin K} J(\underline{k}) > \inf_{\underline{k}\in K} J(\underline{k})$$

Further, since $J(\cdot)$ is lsc on Γ, by Theorem 7-1 there is some set $S^* \subset K$ such that

$$\underline{k}^* \in S^* \Rightarrow J(\underline{k}^*) = \inf_{\underline{k}\in K} J(\underline{k}) = \inf_{\underline{k}\in \Gamma} J(\underline{k})$$

(2) Let

$$d = \inf_{\underline{k}\notin K} J(\underline{k}) - \inf_{\underline{k}\in K} J(\underline{k}) > 0$$

Since $J_n(\cdot) \to J(\cdot)$ uniformly on Γ, there is some N such that if $n \geq N$

$$|J_n(\underline{k}) - J(\underline{k})| \leq d/3 \text{ for all } \underline{k} \in \Gamma$$

Thus, if $\underline{k} \in K$

$$\begin{aligned} J_n(\underline{k}) &\geq J(\underline{k}) - d/3 \\ &\geq \inf_{\underline{k}\notin K} J(\underline{k}) - d/3 \\ &\geq \inf_{\underline{k}\in K} J(\underline{k}) + 2d/3 \\ &= J(\underline{k}^*) + 2d/3 \text{ for some } \underline{k}^* \in S^* \end{aligned}$$

Similarly

$$J_n(\underline{k}^*) \leq J(\underline{k}^*) + d/3$$

$$\Rightarrow \quad J_n(\underline{k}^*) < J_n(\underline{k}) \text{ for any } \underline{k} \notin K$$

Thus

$$\inf_{\underline{k}\in\Gamma} J_n(\underline{k}) = \inf_{\underline{k}\in K} J_n(\underline{k})$$

for all $n \geq N$.

Since $J_n(\cdot)$ is lsc and K is compact, by Theorem 7-1, S_n^* exists such that

$$k_n^* \in S_n^* \Rightarrow J_n(\underline{k}_n^*) = \inf_{\underline{k}\in K} J_n(\underline{k}) = \inf_{\underline{k}\in\Gamma} J_n(\underline{k})$$

(3) Now, since $S_n^* \subset K$ and $S^* \subset K$, K compact, Condition (1) of Theorem 7-4 is satisfied.

Since $J_n(\cdot) \to J(\cdot)$ uniformly on Γ, it follows from Lemma 7-3 that Conditions (2) and (3) of Theorem 7-4 are satisfied also.

Consequently, it follows from Theorem 7-4 that

$$S_n^* \to \hat{S} \subset S^*$$

and that

$$\lim_{n\to\infty} J_n(\underline{k}_n^*) = \lim_{n\to\infty} J(\underline{k}_n^*) = J(\underline{k}^*)$$

where $\underline{k}_n^* \in S_n^*$ and $\underline{k}^* \in S^*$. ■

Since it was the difficulty of dealing directly with the cost function $J(\cdot)$ that motivated the design procedure considered here originally, it is desirable to have an optimality theorem that establishes the results of Theorem 7-6 on the basis of the properties of the approximating functionals $\{J_n(\cdot)\}$. Because of the uniform convergence requirement imposed on the sequence $\{J_n(\cdot)\}$, it is not too difficult to establish such a theorem, on the basis of the following lemma.

Lemma 7-4

If $J_n(\cdot)$ is lsc on the closed set Γ for all n and the sequence $\{J_n(\cdot)\}$ converges to $J(\cdot)$ uniformly on compact subsets of Γ, then $J(\cdot)$ is lsc.

Proof

(1) To prove $J(\cdot)$ is lsc on Γ, it must be shown that given any $\underline{\hat{k}} \in \Gamma$ and any $\xi > 0$, there exists $\eta > 0$ such that

$$\underline{k} \in \Gamma, \ \|\underline{k} - \underline{\hat{k}}\| \ \underline{<} \ \eta \ \Rightarrow \ J(\underline{k}) \ \underline{>} \ J(\underline{\hat{k}}) - \xi$$

(2) To accomplish this, consider the set

$$K = \{\underline{k} \in \Gamma \mid \|\underline{k} - \underline{\hat{k}}\| \ \underline{<} \ 1\}$$

$$= \Gamma \cap \{\underline{k} \mid \|\underline{k} - k\| \ \underline{<} \ 1\}$$

which is compact since Γ is closed.

(3) Thus, $J_n(\cdot) \rightarrow J(\cdot)$ uniformly on K so that there is some integer N such that $n \ \underline{>} \ N \Rightarrow |J_n(\underline{k}) - J(\underline{k})| \ \underline{<} \ \xi/3$ for any $\underline{k} \in K$

(4) Since $J_n(\cdot)$ is lsc on Γ, hence K, there is some $\delta > 0$ such that

$$\underline{k} \in \Gamma, \ \|\underline{k} - \underline{\hat{k}}\| \ \underline{<} \ \delta \ \Rightarrow \ J_N(\underline{k}) \ \underline{>} \ J_N(\underline{\hat{k}}) - \xi/3$$

(5) Thus, let $\eta = \min(1,\delta)$ so that if $\underline{k} \varepsilon \Gamma$, $||\underline{k} - \hat{\underline{k}}|| \leq \eta$ then

$$J(\underline{k}) \geq J_N(\underline{k}) - \xi/3 \quad \text{(by (3))}$$

$$\geq J_N(\hat{\underline{k}}) - 2\xi/3 \quad \text{(by (4))}$$

$$\geq J(\hat{k}) - \xi \quad \text{(by (3))}$$

$\Rightarrow J(\cdot)$ is lsc on Γ. ■

Combining the results of this lemma with those of Theorem 7-5 and the definition of a semicoercive function yields the following optimality theorem.

Theorem 7-7

If the following conditions are met:

(1) Γ is a closed set.

(2) $J_n(\cdot) \to J(\cdot)$ uniformly on compact subsets of Γ.

(3) $J_n(\underline{k}) \geq J_0(\underline{k})$ for all $\underline{k} \varepsilon \Gamma$ for some semicoercive function $J_0(\cdot)$.

(4) $J_n(\cdot)$ is lsc for all n.

then

(5) All global infima of $J_n(\cdot)$ and $J(\cdot)$ occur on some compact set K, i.e., S_n^*, $S^* \subset K$.

(7) $S_n^* \to \hat{S} \subset S^*$ as $n \to \infty$.

(8) $\lim_{n\to\infty} J_n(\underline{k}_n^*) = \lim_{n\to\infty} J(\underline{k}_n^*) = J(\underline{k}^*)$ where $\underline{k}_n^* \varepsilon S_n^*$, $\underline{k}^* \varepsilon S$.

Proof

(1) By Lemma 7-4, since $J_n(\cdot) \to J(\cdot)$ uniformly on compact subsets of Γ and $J_n(\cdot)$ is lsc on Γ, $J(\cdot)$ is lsc on Γ.

(2) Since $J_0(\cdot)$ is semicoercive

$$\lim_{||\underline{k}||\to\infty} \frac{J_0(\underline{k})}{||\underline{k}||} = \alpha$$

for some $\alpha > 0$. In particular, there is some $\beta > 0$ such that $||\underline{k}|| > \beta \Rightarrow J_0(\underline{k}) \geq \alpha/2 \; ||\underline{k}||$.

(3) Now, pick any $\underline{k}_0 \in \Gamma$ and let

$$\beta' = \max\left(\beta, \frac{4J(\underline{k}_0)}{\alpha}, ||\underline{k}_0||\right)$$

so that if $||\underline{k}|| > \beta'$ then

$$J_0(\underline{k}) \geq \frac{\alpha}{2} ||\underline{k}|| > \frac{\alpha\beta'}{2} \geq 2J(\underline{k}_0)$$

(4) Next, define $K = \{\underline{k} \in \Gamma \,|\, ||\underline{k}|| \leq \beta'\}$ and note that for any n

$$\inf_{\underline{k}\notin K} J_n(\underline{k}) = \inf_{||\underline{k}||>\beta'} J_n(\underline{k}) \geq \inf_{||\underline{k}||>\beta'} J_0(\underline{k}) \geq 2J(\underline{k}_0)$$

Further, since $J_n(\underline{k}_0) \to J(\underline{k}_0)$, there is some integer N such that

$$n > N \Rightarrow |J_n(\underline{k}_0) - J(\underline{k}_0)| \leq \frac{1}{2} J(\underline{k}_0) \Rightarrow J_n(\underline{k}_0) \leq \frac{3}{2} J(\underline{k}_0)$$

Thus

$$\inf_{\underline{k}\notin K} J_n(\underline{k}) > J_n(\underline{k}_0) \geq \inf_{\underline{k}\in\Gamma} J_n(\underline{k})$$

so that $S_n^* \subset K$ for all n.

(5) Similarly, since $J(\cdot)$ is lsc and K is compact, $J(\cdot)$ achieves its global infimum over K on some set $S^* \subset K$. Also, for any $\underline{k} \notin K$

$$J(\underline{k}) = \lim_{n\to\infty} J_n(\underline{k}) \geq 2J(\underline{k}_0) > J(k_0) \geq \inf_{\underline{k}\in\Gamma} J(\underline{k})$$

so that all global minima of $J(\cdot)$ occur on K.

(6) Consequently, Condition (5) is established, and Conditions (6) and (7) follow from Theorem 7-6. ∎

7.5 Summary of Convergence Conditions

The results presented in this section, culminating in Theorem 7-7, provide sufficient conditions on the set Γ of admissible compensator designs and the approximating costs $\{J_n(\cdot)\}$ to guarantee that the four fundamental questions raised in the introduction have (approximately) affirmative answers. Specifically, these conditions guarantee the following results:

(1) A set S^* of optimal compensator designs for the distributed parameter plant P exists.

(2) For every n, a set S_n^* of optimal compensator designs for the finite dimensional approximating plant P_n exists.

(3) The sequence $\{S_n^*\}$ of sets of optimal designs for the plants $\{P_n\}$ converges to a set S of optimal designs for the distributed parameter plant P.

Several important points should be noted regarding these results. First the assumption that the set Γ of admissible compensators is closed is essential. In particular, this requirement is imposed to avoid the possibility that $\inf_{\underline{k}\in\Gamma} J(k)$ exists but is not achieved on Γ. A simple illustration of this is the following: minimize $J(k) = -k$ over $\Gamma = (-\infty,0)$. Here

$$\inf_{k\in\Gamma} J(k) = 0$$

even though $J(k) > 0$ for all $k \in \Gamma$.

This example is unrealistic in certain respects, however, since it is to be expected that the definition of Γ will somehow reflect a stabilizability requirement and that $J(\underline{k})$ will approach $+\infty$ as $\underline{k}$ approaches the boundary of Γ. A more representative example is therefore the following: minimize

$$J(\underline{k}) = k_2^2 + \left[\frac{k_1^2 - k_2^2}{k_1 k_2}\right] \text{ over } \Gamma = \{\underline{k} = (k_1,k_2) \mid -\infty < k_1,k_2 < 0\}$$

Here, for any fixed k_1, $J(\underline{k}) \to +\infty$ as $k_2 \to 0^-$ and similarly for fixed k_2, $J(\underline{k}) \to +\infty$ as $k_1 \to 0^-$. Further, this function is coercive, suggesting it should achieve its infimum over Γ on some compact set $C \subset \Gamma$. Note, however, that for any fixed k_2

$$\inf_{k_1 \varepsilon (-\infty,0)} J(k_1,k_2) = k_2^2$$

when $k_1 = k_2$. Thus, the sequence $\{\underline{k}_n\} = \{(-1/n,-1/n)\}$ yields the sequence of costs $\{J(\underline{k}_n)\} = \{1/n^2\}$ which converges to 0 as $n \to \infty$. However, for any $\underline{k} \varepsilon \Gamma$, $J(\underline{k}) \geq k_2^2 > 0$, $\inf_{\underline{k}\varepsilon\Gamma} J(\underline{k}) = 0$ is never achieved since the minimizing sequence $\{(-1/n,-1/n)\}$ converges to a limit outside Γ. Consequently, since the set Λ of stabilizing compensators meeting any particular structural constraint is usually an open set (corresponding to the fact that the root locus is continuous), it is clear that Γ must be taken as some closed subset of Λ. Physically, this corresponds to the imposition of some minimum stability margin or some other standard of acceptability like that considered in Reference 7-9. Another possibility would be to restrict the optimization a priori by placing explicit bounds on each component k_i of the parameter vector $\underline{k}$. Such a restriction would further guarantee that the set Γ is compact, which makes the results of Theorem 7-5 directly applicable.

The uniform convergence requirement on the sequence $\{J_n(\cdot)\}$ is also extremely important, as illustrated by the example in Figure 7-2. Here, $J_n^*(k) \to J(k)$ for all $k \varepsilon R^1$ but $k^* = 2$ and $J(k^*) = 1$ while $k_n^* = 1/2n$ and $\{J_n(k_n^*)\} = 0$ for all n. Thus, the sequences $\{k_n^*\}$ and $\{J_n(k_n^*)\}$ both converge to well defined limits that are not k^* and $J(k^*)$, respectively.

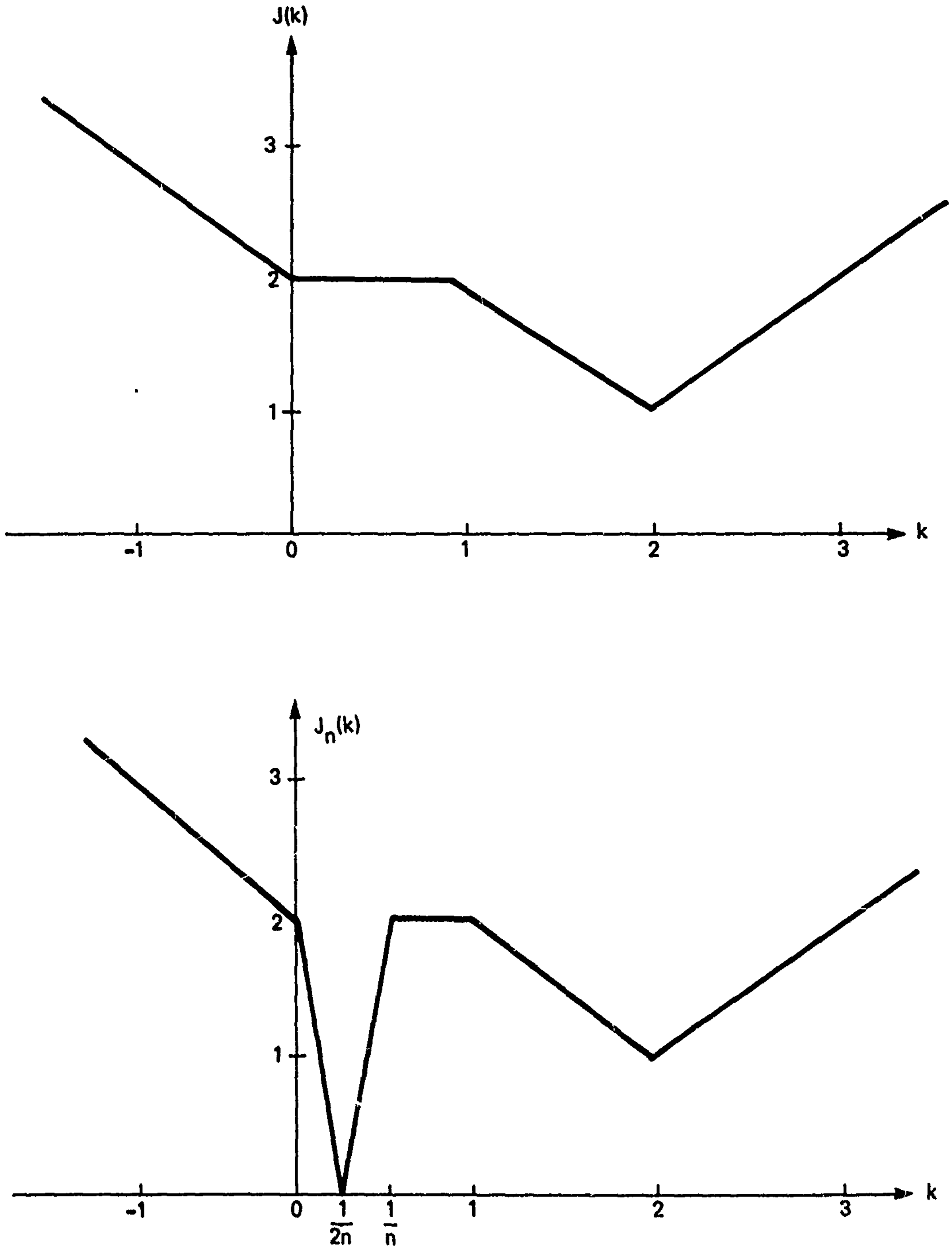

Figure 7-2.

The requirement that $J_n(\cdot)$ be lsc for all n is not particularly restrictive. Specifically, in most practical compensator design problems, the cost functionals are chosen to be differentiable so that gradient-derived necessary conditions for optimality or gradient-based computational algorithms may be used to compute the optimal compensator parameters. Consequently, $J_n(\cdot)$ will normally be continuous and thus lsc.

Assuming all of these conditions are met, it is important to note that they only guarantee the convergence of the sequence of sets $\{S_n^*\}$ <u>into</u> the set S^*. If S^* consists of a unique optimal design $\underline{k}^*$, then any sequence $\{k_n\}$ of elements of $\{S_n^*\}$ is guaranteed to converge to $\underline{k}^*$ if the conditions of Theorems 7-5, 7-6 or 7-7 are met. If, however, S^* contains more than one optimal design, then the limit of the sequence $\{S_n^*\}$ need be only a subset of S^*. This point is illustrated by the following example:

$$J(k) = \begin{cases} -k-1 & , \quad k \leq -1 \\ k+1 & , \quad -1 < k \leq 0 \\ 1-k & , \quad 0 < k \leq 1 \\ k-1 & , \quad k > 1 \end{cases}$$

$$J_n(k) = \begin{cases} -k-1-\frac{1}{n} & , \quad k \leq -1 \\ \left(1+\frac{1}{n}\right) k + 1 & , \quad -1 < k < 0 \\ 1 - \left(1+\frac{1}{2n}\right) k & , \quad 0 < k \leq 1 \\ k-1-\frac{1}{2n} & , \quad k > 1 \end{cases}$$

Note that only $|J_n(k) - J(k)| \leq 1/n$ for all $k \in R^1$ so $J_n(\cdot) \to J(\cdot)$ uniformly on R^1, but while J(k) exhibits minima at $k = \pm 1$, $J_n(k)$ exhibits a unique global minimum at $k = -1$ for all n. Thus, $S_n^* = \{-1\}$, $S^* = \{-1,1\}$, so that $S_n^* \to \hat{S} \subset S^*$ but $\hat{S} \neq S^*$.

Finally, note that the convergence conditions developed here are only sufficient and not necessary. In fact, it would appear to be very difficult to develop any necesary conditions for convergence, as the following example illustrates.

Suppose $J(k) = |k|$ and take

$$J_n(k) = \begin{cases} (1 + \varepsilon + \frac{1}{n}) \; |k| & , \text{ if n odd} \\ (1 - \varepsilon + \frac{1}{n}) \; |k| & , \text{ if n even} \end{cases}$$

for some $\varepsilon > 0$. Clearly both $J(k)$ and $J_n(k)$ exhibit unique global minima at $k = 0$ for which $J(k) = J_n(k) = 0$. Thus, $k_n^* \to k^*$ and $J_n(k_n^*) \to J(k^*)$, but the sequence of functions $\{J_n(\cdot)\}$ does not converge to anything anywhere except at $k = 0$.

7.6 Conclusions

The results developed are general enough to apply to a wide variety of fixed-form compensator design problems. Current efforts are aimed at specializing them to the optimal output feedback compensator design problem that originally motivated these efforts. In particular, the following questions are currently under active investigation.

(1) How can the conditions of Theorem 7-7 be interpreted physically? It is possible that approximation results for strongly continuous semigroups can be used to guarantee that these conditions are satisfied for output feedback control of flexible structures, subject to certain conditions on the inherent damping in the structure.

(2) For problems in which the convergence conditions are met, what factors influence the rate of convergence? This question is of considerable practical importance because the rate of convergence determines how many modes must be retained in a finite element model of a structure to provide a reasonable basis for control system design.

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SECTION 8

LARGE-ANGLE SPACECRAFT SLEWING MANEUVERS

8.1 Introduction

The problem of large-angle maneuvers for flexible spacecraft continues to be of interest in the aerospace community. This section develops a number of extensions to the work first reported in Reference 8-1. In particular, this section presents techniques for improving the optimal torque profiles by

(1) Allowing the solution process to determine the optimal terminal boundary conditions for the maneuvers.

(2) Developing a control-rate penalty technique for producing smooth control profiles.

Several example maneuvers are provided that demonstrate the practical application and utility of the techniques developed herein.

This section is presented in five parts. Section 8.2 treats the distributed control problem and develops the techniques for handling slewing maneuvers when the final angle for the maneuver is to be determined as part of the solution. The results of Section 8.2 are appropriate for maneuvers where the final angular rate, rather than the final orientation, is important.

Section 8.3 treats the optimal distributed control problem and develops a control-rate penalty technique for smoothing the optimal control torque profiles. The control designer gains the ability to specify the terminal control state and control rates as a by-product of the control-rate penalty technique. In addition, Section 8.3 combines the free final angle problem of Section 8.2 with the control-rate penalty technique.

Section 8.4 develops techniques for solving a single-axis retargeting maneuver for a rigid spacecraft. Computational algorithms are proposed for handling systems modeled by nonlinear dynamics and time-varying weighting matrices in the performance index.

Section 8.5 presents the results of a preliminary study of slewing maneuvers for the ACOSS Model 2. All example maneuvers are presented in Section 8.6.

8.2 Free Final Angle Optimal Slewing Maneuvers

The specific model considered (see Figure 8-1) consists of a rigid hub with four identical elastic appendages attached symmetrically about the central hub. A single-axis maneuver with the flexible members restricted to displacements in the plane normal to the axis of rotation is the only case considered. Furthermore, the body (as a whole) is assumed to experience only antisymmetric deformation modes (see Figure 8-2). The distributed control system for the vehicle consists of

(1) A single controller in the rigid part of the structure.

(2) Four appendage controllers, one assumed to be located halfway along the span of each of the four elastic appendages.

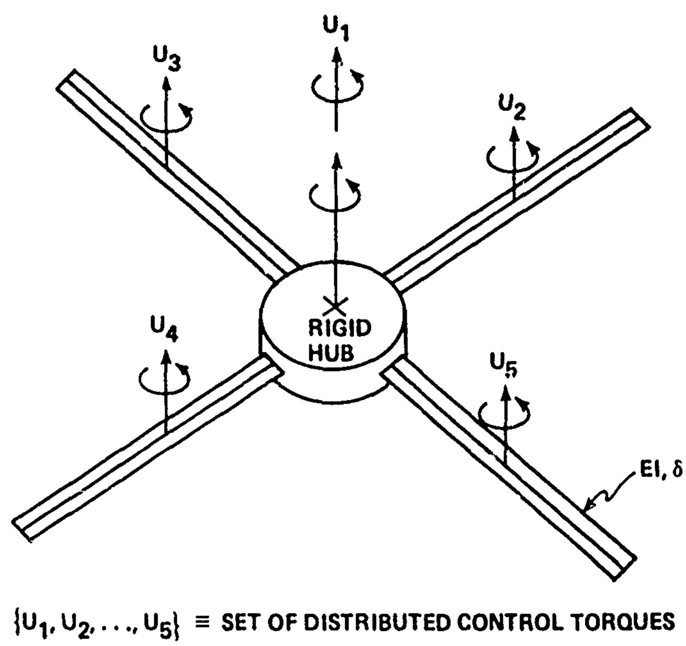


Figure 8-1. Undeformed structure.

The extension to the case of multiple controllers along each appendage is straightforward; however, in Section 8.2 only the single appendage-controller case is presented.

The formulas for large-angle maneuvers presented in this section are developed in physical space, because the free final angle transversality conditions are understood most easily in terms of physical coordinates.

8.2.1 Equations of Motion

For the vehicle shown in Figure 8-1, the equations of motion can be obtained from Hamilton's extended principle [8-2], leading to

$$(\hat{I} - \underline{\eta}^T M^* \underline{\eta})\ddot{\theta} + M_{\theta\eta}^T \ddot{\underline{\eta}} - 2\dot{\theta}\dot{\underline{\eta}}^T M^* \underline{\eta} = u_1 + 4u_2 \qquad (8\text{-}1)$$

$$\underline{M}_{\theta\eta}\ddot{\theta} + M_{\eta\eta}\ddot{\underline{\eta}} + [K_{\eta\eta} + \dot{\theta}^2 M^*]\underline{\eta} = \underline{F}u_2 \qquad (8\text{-}2)$$

where

$$M^* = M - M_{\eta\eta}$$

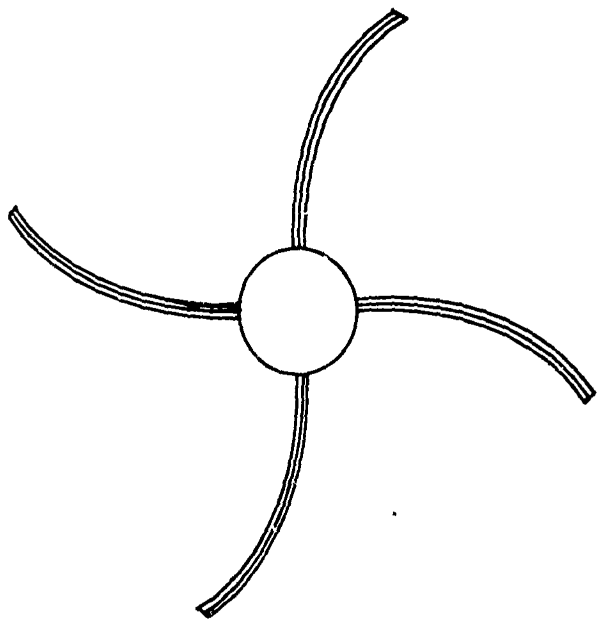


Figure 8-2. Antisymmetric deformation.

8.2.2 State Space Formulation

Referring to Eq. (8-15) in Reference 8-3, the linear time-invariant form of Eq. (8-1) and (8-2) follows as

$$M\underline{\ddot{\zeta}} + K\underline{\zeta} = P\underline{u} \tag{8-3}$$

where

$$M = \begin{bmatrix} I & \underline{M}_{\theta\eta}^T \\ \underline{M}_{\theta\eta} & M_{\eta\eta} \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & \underline{0}^T \\ \underline{0} & K_{\eta\eta} \end{bmatrix}$$

$$P = \begin{bmatrix} \underline{1} & \underline{4} \\ \underline{0} & \underline{F} \end{bmatrix}$$

$$\underline{\zeta} = [\theta \ \underline{n}^T]^T$$

$$\underline{u} = [u_1 \ u_2]^T$$

$$M = M^T \text{ (positive definite)}$$

$$K = K^T \text{ (positive semidefinite)}$$

Defining the state variable subsets as

$$\underline{\Sigma}_1 = \underline{\zeta} \qquad\qquad \underline{\Sigma}_2 = \dot{\underline{\zeta}} \tag{8-4}$$

leads to the first-order differential equations

$$\dot{\underline{\Sigma}}_1 = \underline{\Sigma}_2 \qquad\qquad \dot{\underline{\Sigma}}_2 = -M^{-1}K\underline{\Sigma}_1 + M^{-1}P\underline{u} \tag{8-5}$$

Letting $\underline{\Sigma} = [\underline{\Sigma}_1^T \ \underline{\Sigma}_2^T]^T$, the state space equation becomes

$$\dot{\underline{\Sigma}} = A\underline{\Sigma} + B\underline{u} \tag{8-6}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ M^{-1}P \end{bmatrix}$$

8.2.3 Fixed Time Free Final Angle Optimal Control Problem

8.2.3.1 Statement of Problem

The rotational dynamics of a flexible space vehicle, restricted to a single-axis large-angle maneuver, are considered where the system dynamics are governed by Eq. (8-6). The optimal control problem is to find the solution to Eq. (8-6) that minimizes the performance index

$$J = \frac{1}{2} \int_{t_0}^{t_f} [\underline{u}^T W_{uu} \underline{u} + \underline{\Sigma}^T W_{ss} \underline{\Sigma}]\, dt \tag{8-7}$$

and satisfies the specified terminal states, where W_{uu} is the control weighting matrix and W_{ss} is the state weighting matrix. The specified terminal states are given by

$$\underline{\zeta}_0 = [\theta_0 \quad \underline{n}^T(t_0)]^T \qquad \dot{\underline{\zeta}}_0 = [\dot{\theta}_0 \quad \dot{\underline{n}}^T(t_0)]^T \tag{8-8}$$

$$\underline{n}_f = \underline{n}(t_f) \qquad \dot{\underline{\zeta}}_f = [\dot{\theta}_f \quad \dot{\underline{n}}^T(t_f)]^T \tag{8-9}$$

where the constraint

$$\underline{n}(t_f) = \dot{\underline{n}}(t_f) = \underline{0}$$

is imposed in Eq. (8-9). In addition, the final angle θ_f is not specified since it is to be determined as part of the solution.

8.2.3.2 Derivation of the Necessary Conditions from Pontryagin's Principle

In preparing to use Pontryagin's necessary conditions, the Hamiltonian functional

$$H = \frac{1}{2}(\underline{u}^T W_{uu} \underline{u} + \underline{\Sigma}^T W_{ss} \underline{\Sigma}) + \underline{\Lambda}^T (A\underline{\Sigma} + B\underline{u}) \tag{8-10}$$

is introduced, where the $\underline{\Lambda}$'s are Lagrange multipliers (also known as costate or adjoint variables). Pontryagin's principle requires, as a necessary condition, that the $\underline{\Lambda}$'s satisfy costate differential equations derivable from

$$\underline{\dot{\Lambda}} = -\frac{\partial H}{\partial \underline{s}} = -W_{ss}\underline{\Sigma} - A^T\underline{\Lambda} \tag{8-11}$$

and that the control torque $\underline{u}$ must be chosen at every instant so that the Hamiltonian of Eq. (8-10) is minimized; that is, for continuous $\underline{u}$ one requires that

$$\frac{\partial H}{\partial \underline{u}} = \underline{0} = W_{uu}\underline{u} + B^T\underline{\Lambda} \tag{8-12}$$

from which the optimal torque is determined as a function of the costate variables as

$$\underline{u} = -W_{uu}^{-1} B^T\underline{\Lambda} \tag{8-13}$$

Substituting Eq. (8-13) into Eq. (8-6) yields the state differential equation

$$\underline{\dot{\Sigma}} = A\underline{\Sigma} - BW_{uu}^{-1}B^T\underline{\Lambda} \tag{8-14}$$

Because the final angle is assumed to be free, the transversality condition providing the natural boundary condition for the problem follows as

$$-\underline{\Lambda}^T(t_f)\delta\underline{\Sigma}_f = 0 \tag{8-15}$$

where $\underline{\Lambda}(t)$ is the costate vector in physical space and $\delta\underline{\Sigma}_f$ is the variation of the physical space state at the final time. Since θ_f is free and $\dot{\theta}_f$, $\underline{\eta}_f$, and $\underline{\dot{\eta}}_f$ are specified, $\delta\underline{\Sigma}_f$ in Eq. (8-15) can be written as

$$\delta\underline{\Sigma}_f = [\delta\theta_f \; \delta\underline{\eta}_f^T \; \delta\dot{\theta}_f \; \delta\underline{\dot{\eta}}_f^T]^T = [\delta\theta_f \; \underline{0}^T \; 0 \; \underline{0}^T]^T \tag{8-16}$$

Upon introducing Eq. (8-16) into Eq. (8-15), the desired free final angle transversality condition follows as

$$\Lambda_1(t_f)\,\delta\theta_f = 0$$

or

$$\Lambda_1(t_f) = 0 \tag{8-17}$$

since $\delta\theta_f$ is arbitrary, where

$$\underline{\Lambda} = [\Lambda_1 \quad \Lambda_2 \quad \dots \quad \Lambda_{2N}]^T \quad (N = n + 1)$$

and Λ_1 denotes the costate variable for the rigid-body rotation angle θ.

Thus, $\Lambda_1(t_f)$, $\dot{\theta}_f$, $\underline{\eta}_f$, and $\dot{\underline{\eta}}_f$ provide the 2N final time boundary conditions necessary for specifying the optimal control solution of Eq. (8-11) and (8-14), when fixed-time free final angle maneuvers are of interest.

8.2.3.3 Solution for the Initial Costates

In Eq. (8-11) and (8-14), the initial boundary conditions for $\underline{\Sigma}$ are completely known while for $\underline{\Lambda}$ they are unknown, and the final boundary conditions for $\underline{\Sigma}$ and $\underline{\Lambda}$ are only partially known. Thus, the application of Pontryagin's principle has led, as usual, to a two-point boundary-value problem (TPBVP). To solve Eq. (8-11) and (8-14), the merged state vector is defined as

$$\underline{X} = [\underline{\Sigma}^T \quad \underline{\Lambda}^T]^T \tag{8-18}$$

from which it follows that

$$\dot{\underline{X}} = \Omega\underline{X} \tag{8-19}$$

where

$$\Omega = \begin{bmatrix} A & -BW_{uu}^{-1}B^T \\ -W_{ss} & -A^T \end{bmatrix}$$

= constant coefficient matrix

Since Ω is constant, it is well known that Eq. (8-19) possesses the solution

$$\underline{X}(t) = e^{\Omega(t-t_0)} \underline{X}(t_0) \tag{8-20}$$

where $e^{\Omega(t-t_0)}$ is the $4N \times 4N$ exponential matrix.

The solution for $e^{\Omega(t-t_0)}$ is obtained conveniently by a variety of methods discussed in either Reference 8-3 or 8-4. Upon setting

$$e^{\Omega(t-t_0)} = \Phi(t,t_0)$$

and writing Eq. (8-20) in partitioned form, one obtains

$$\begin{Bmatrix} \underline{\Sigma}(t_f) \\ \underline{\Lambda}(t_f) \end{Bmatrix} = \begin{bmatrix} \Phi_{\Sigma\Sigma} & \Phi_{\Sigma\Lambda} \\ \Phi_{\Lambda\Sigma} & \Phi_{\Lambda\Lambda} \end{bmatrix} \begin{Bmatrix} \underline{\Sigma}(0) \\ \underline{\Lambda}(0) \end{Bmatrix} \tag{8-21}$$

or equivalently

$$\begin{Bmatrix} \Sigma_1(t_f) \\ \vdots \\ \Sigma_{2N}(t_f) \\ \Lambda_1(t_f) \\ \vdots \\ \Lambda_{2N}(t_f) \end{Bmatrix} = \begin{bmatrix} \Phi_{\Sigma_1\Sigma_1} & \cdots & \Phi_{\Sigma_1\Sigma_{2N}} & \Phi_{\Sigma_1\Lambda_1} & \cdots & \Phi_{\Sigma_1\Lambda_{2N}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{\Sigma_{2N}\Sigma_1} & \cdots & \Phi_{\Sigma_{2N}\Sigma_{2N}} & \Phi_{\Sigma_{2N}\Lambda_1} & \cdots & \Phi_{\Sigma_{2N}\Lambda_{2N}} \\ \Phi_{\Lambda_1\Sigma_1} & \cdots & \Phi_{\Lambda_1\Sigma_{2N}} & \Phi_{\Lambda_1\Lambda_1} & \cdots & \Phi_{\Lambda_1\Lambda_{2N}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{\Lambda_{2N}\Sigma_1} & \cdots & \Phi_{\Lambda_{2N}\Sigma_{2N}} & \Phi_{\Lambda_{2N}\Lambda_1} & \cdots & \Phi_{\Lambda_{2N}\Lambda_{2N}} \end{bmatrix} \begin{Bmatrix} \Sigma_1(0) \\ \vdots \\ \Sigma_{2N}(0) \\ \Lambda_1(0) \\ \vdots \\ \Lambda_{2N}(0) \end{Bmatrix} \tag{8-22}$$

Observing that the known quantities in Eq. (8-22) are $\Sigma_2(t_f),\ldots,\Sigma_{2N}(t_f)$, $\Lambda_1(t_f)$, and $\Sigma_1(0),\ldots,\Sigma_{2N}(0)$ and carrying out the partitioned matrix multiplication to solve for $\Sigma_2(t_f),\ldots,\Sigma_{2N}(t_f)$, $\Lambda_1(t_f)$, the equation defining the solution for $\underline{\Lambda}(0)$ follows as

$$[A]\underline{\Lambda}(0) = \underline{b} - [B]\underline{\Sigma}(0) \qquad (8\text{-}23)$$

where

$$[A] = \begin{bmatrix} \phi_{\Sigma_2\Lambda_1} & \cdots & \phi_{\Sigma_2\Lambda_{2N}} \\ \vdots & & \vdots \\ \phi_{\Sigma_{2N}\Lambda_1} & \cdots & \phi_{\Sigma_{2N}\Lambda_{2N}} \\ \phi_{\Lambda_1\Lambda_1} & \cdots & \phi_{\Lambda_1\Lambda_{2N}} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \phi_{\Sigma_2\Sigma_1} & \cdots & \phi_{\Sigma_2\Sigma_{2N}} \\ \vdots & & \vdots \\ \phi_{\Sigma_{2N}\Sigma_1} & \cdots & \phi_{\Sigma_{2N}\Sigma_{2N}} \\ \phi_{\Lambda_1\Sigma_1} & \cdots & \phi_{\Lambda_1\Sigma_{2N}} \end{bmatrix}$$

$$\underline{b} = [\Sigma_2(t_f) \ \ldots \ \Sigma_{2N}(t_f) \ \Lambda_1(t_f)]^T$$

Equation (8-23) is in the standard form for the linear algebraic equation $A\underline{x} = \underline{b}$, which can be solved via Gaussian elimination for $\underline{x}$ to yield $\underline{\Lambda}(0)$.

The optimal control time histories are obtained upon integrating Eq. (8-11) and (8-14) subject to $\underline{\Sigma}(0)$ given by Eq. (8-8) and $\underline{\Lambda}(0)$ given by Eq. (8-23). Equation (8-23) provides the initial costates in physical coordinates. However, if the maneuver simulations are modeled in modal coordinates, the following two transformations permit the use of the results of this section.

First, the modal space state transition matrix, $\phi(t_f,0)$ is mapped to physical space via the transformation

$$\Phi(t_f,0) = \Theta\hat{\Phi}(t_f,0)\Theta^{-1} \tag{8-24}$$

where

$$\Theta = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & ME & 0 \\ 0 & 0 & 0 & ME \end{bmatrix}$$

$$\Theta^{-1} = \begin{bmatrix} E^TM & 0 & 0 & 0 \\ 0 & E^TM & 0 & 0 \\ 0 & 0 & E^T & 0 \\ 0 & 0 & 0 & E^T \end{bmatrix}$$

The derivation for the transformation matrix, Θ, can be found in either Reference 8-5 or 8-6.

Second, introducing the required partitions of Eq. (8-24) into Eq. (8-23) and solving for $\underline{\Lambda}(0)$, the modal space initial conditions are obtained from the transformation equation

$$\begin{Bmatrix} \underline{s}(0) \\ \underline{\lambda}(0) \end{Bmatrix} = \Theta^{-1} \begin{Bmatrix} \underline{\Sigma}(0) \\ \underline{\Lambda}(0) \end{Bmatrix} \tag{8-25}$$

where $\underline{\lambda}(0)$ is the modal space initial costate. The optimal control time histories for modal space are obtained upon integrating the modal space form of Eq. (8-11) and (8-14) subject to the initial conditions provided by Eq. (8-25).

Case 1 in Section 8.6 provides an example maneuver.

8.3 Control-Rate Penalty Technique for Optimal Slewing Maneuvers

Introducing control-rate penalties into the performance index accomplishes two things. First, the jump discontinuities in the control time history are pushed into the higher control time derivatives. In other words, the control-rate penalties act like control smoothing penalties. As a result, the frequency content of the resulting control profile is significantly reduced; thus, the vehicle's higher frequency modes are excited only mildly by the smoothed control profile.

Second, the use of control-rate penalties permits the control designer to specify the terminal control state and control time derivatives. For example, in many reasonable maneuvers, it is convenient to specify that the slewing control system is turned off both initially and finally. Of course, depending on particular mission objectives (as shown in the retargeting maneuvers of Section 8.4), other terminal boundary conditions for the control system are possible. To fully appreciate the significant gain in performance that the control-rate penalty technique permits, compare the example maneuvers of Section 8.6 with the results presented in Section 8 of Reference 8-1.

The vehicle depicted in Figure 8-1 is assumed to be modeled for the results of this section. Only the case of a single-axis maneuver is considered, where the elastic displacements occur in the plane normal to the axis of rotation, and only antisymmetric modes are modeled (see Figure 8-2).

8.3.1 Problem Formulation

From Eq. (8-3), the equations of motion can be written as

$$M\ddot{\underline{\zeta}} + K\underline{\zeta} = P\underline{u} \tag{8-26}$$

As shown in Reference 8-1, Eq. (8-26) can be cast into state space form, through transformation sequences ($\underline{\zeta} \rightarrow \underline{t} \rightarrow \underline{s}_1$) and ($\underline{s}_1, \underline{s}_2 \rightarrow \underline{s}$), leading to

$$\dot{\underline{s}} = A\underline{s} + B\underline{u} \tag{8-27}$$

The slewing maneuver problem is to find the solution to Eq. (8-27) that satisfies the terminal boundary conditions

$$\underline{\zeta}_0 = [\theta_0 \; \underline{n}^T(t_0)]^T \quad \dot{\underline{\zeta}}_0 = [\dot{\theta}_0 \; \dot{\underline{n}}^T(t_0)]^T \quad \underline{u}_0^{(i)} = \underline{u}^{(i)}(t_0) \tag{8-28}$$

$$\underline{\zeta}_f = [\theta_f \; \underline{n}^T(t_0)]^T \quad \dot{\underline{\zeta}}_f = [\dot{\theta}_f \; \dot{\underline{n}}^T(t_f)]^T \quad \underline{u}_f^{(i)} = \underline{u}^{(i)}(t_f) \tag{8-29}$$

for $i = 0,1,\ldots,k-1$, where

$$\underline{n}(t_f) = \dot{\underline{n}}(t_f) = \underline{0}$$

$$\underline{u}^{(i)}(t_0) = \underline{u}^{(i)}(t_f) = \underline{0}$$

with

$$\underline{u}^{(0)} = \underline{u}$$

$$\underline{u}^{(i)} = \frac{d^i}{dt^i}(\underline{u})$$

The optimal control problem is to seek the torque history $\underline{u}(t)$ that will generate an optimal solution of Eq. (8-27), initiating at Eq. (8-28) and terminating at Eq. (8-29), that minimizes the performance index

$$J = \frac{1}{2}\int_0^{t_f} \left(\underline{s}^T W_{ss} \underline{s} + \sum_{i=0}^{k} \underline{u}^{(i)T} W_{ii} \underline{u}^{(i)}\right) dt \tag{8-30}$$

where W_{ss} is a weighting matrix for the states, W_{00} is a weighting matrix for the control, and W_{ii} for $i = 1,\ldots,k$ are weighting matrices for the control rates. With k equal to zero, the performance index penalizes only the states and controls, and thus the control problem corresponds to a problem of minimum control energy, kinetic energy, and elastic potential energy. The solution for this problem was addressed in Reference 8-1. With k greater than zero, the performance index penalizes the states, controls, and time derivatives of the controls. The example maneuvers in Section 8.6 demonstrate that including control-rate penalities leads to much smoother modal amplitude and control torque time histories. Additional example maneuvers can be found in Reference 8-6. The flexible-body response is improved because the vehicle's flexural degrees of freedom are very sensitive to discontinuities in the control time histories; hence, by using the formulation in this section, the jump discontinuities in the control at $t = 0$ and $t = t_f$ are moved to the higher control time derivatives, which improves overall system performance.

8.3.2 Necessary Conditions for the Optimal Control Problem

Adjoining the equation of motion, Eq. (8-27), as a differential equation constraint to the performance index in Eq. (8-30), the augmented performance index is given by

$$\hat{J} = \int_0^{t_f} \left\{\frac{1}{2}\left[\underline{s}^T W_{ss}\underline{s} + \sum_{i=0}^{k} \underline{u}^{(i)T} W_{ii}\underline{u}^{(i)}\right] + \underline{\lambda}^T\left[A\underline{s} + B\underline{u} - \dot{\underline{s}}\right]\right\} dt \tag{8-31}$$

where the $\underline{\lambda}$'s are Lagrange multipliers. Using standard calculus of variations techniques the first variation of $\hat{J}$ follows as

$$\delta\hat{J} = \int_0^{t_f} \left\{\delta\underline{s}^T W_{ss}\underline{s} + \sum_{i=0}^{k} \delta\underline{u}^{(i)T} W_{ii}\underline{u}^{(i)} + \delta\underline{\lambda}^T(A\underline{s} + B\underline{u} - \dot{\underline{s}})\right.$$

$$\left. + \underline{\lambda}^T(A\delta\underline{s} + B\delta\underline{u} - \delta\dot{\underline{s}})\right\} dt \tag{8-32}$$

Integrating the terms containing $\delta\dot{\underline{s}}$ and $\delta\underline{u}^{(i)}$ by parts for $i = 1,\ldots,k$ leads to

$$-\int_0^{t_f} \underline{\lambda}^T\delta\dot{\underline{s}}\, dt = -\underline{\lambda}^T\delta\underline{s}\Big|_0^{t_f} + \int_0^{t_f} \dot{\underline{\lambda}}^T\delta\underline{s}\, dt \tag{8-33}$$

and

$$\int_0^{t_f} \delta\underline{u}^{(i)T} W_{ii}\underline{u}^{(i)}\, dt = \sum_{j=1}^{i} (-1)^{j-1} \delta\underline{u}^{(i-j)T} W_{ii}\underline{u}^{(i+j-1)} \Big|_0^{t_f}$$

$$+ (-1)^i \int_0^{t_f} \delta\underline{u}^T W_{ii}\underline{u}^{(2i)}\, dt \qquad (8\text{-}34)$$

since $\underline{s}$ and $\underline{u}^{(i)}$ (for $i = 0,1,\ldots,k-1$) are specified both initially and finally in Eq. (8-28), the terminal variational terms in Eq. (8-33) and (8-34) are

$$\delta\underline{s}(t_0) = \delta\underline{s}(t_f) = \underline{0}$$

$$\delta\underline{u}^{(i-j)}(t_0) = \delta\underline{u}^{(i-j)}(t_f) = \underline{0} \quad (j = 1,\ldots,i)$$

$$(8\text{-}35)$$

where $t_0 = 0$. Substituting Eq. (8-33), (8-34), and (8-35) into Eq. (8-32) yields

$$\delta\hat{J} = \int_0^{t_f} \Big\{\Big[\sum_{i=0}^{k} (-1)^i W_{ii}\underline{u}^{(2i)} + B^T\underline{\lambda}\Big]^T \delta\underline{u} + \Big[W_{ss}\underline{s} + A^T\underline{\lambda} + \dot{\underline{\lambda}}\Big]^T \delta\underline{s}$$

$$+ \Big[A\underline{s} + B\underline{u} - \dot{\underline{s}}\Big]^T \delta\underline{\lambda}\Big\}\, dt \qquad (8\text{-}36)$$

Since Pontryagin's principle requires the first variation of $\hat{J}$ to be zero, the necessary conditions follow as

$$\dot{\underline{s}} = A\underline{s} + B\underline{u} \qquad (8\text{-}37)$$

$$\dot{\underline{\lambda}} = -W_{ss}\underline{s} - A^T\underline{\lambda} \qquad (8\text{-}38)$$

$$\sum_{i=0}^{k} (-1)^i W_{ii}\underline{u}^{(2i)} = -B^T\underline{\lambda} \qquad (8\text{-}39)$$

In order to cast Eq. (8-39) into the standard space form, the control state is defined as

$$\underline{U} = [\underline{u}_1^T \quad \underline{u}_2^T \quad \cdots \quad \underline{u}_{2k}^T]^T \tag{8-40}$$

where

$$\underline{u}_i = \underline{u}^{(i-1)} \quad (i = 1,2,\ldots,2k)$$

and $\underline{u}_i$ is an $N_c \times 1$ vector for $i = 1,2,\ldots,2k$, $\underline{U}$ is a $2kN_c \times 1$ vector, and N_c denotes the number of independent controls. Referring to Eq. (8-40), Eq. (8-39) can be written as

$$\dot{\underline{U}} = C\underline{U} + D\underline{\lambda} \tag{8-41}$$

where

$$C = \begin{bmatrix} 0 & I \\ C_{21} & C_{22} \end{bmatrix}$$

$$C_{21} = (-1)^{k+1} W_{kk}^{-1} W_{00}$$

$$C_{22} = (-1)^{k+2} W_{kk}^{-1} [0 \vdots W_{11} \vdots 0 \vdots -W_{22} \vdots 0 \vdots \cdots \vdots 0 \vdots (-1)^k W_{k-1,k-1} \vdots 0]$$

and

$$D = \begin{bmatrix} 0^* \\ (-1)^{k+1} W_{kk}^{-1} B^T \end{bmatrix}$$

where 0^* is a $(2k - 1)N_c \times 2N$ null matrix

Defining the E matrix as follows

$$E = [B \ (0^*)^T] \tag{8-42}$$

and substituting Eq. (8-40) and (8-42) into Eq. (8-37), the state, costate, and control differential equations follow as

State equations

$$\dot{\underline{s}} = A\underline{s} + E\underline{U} \tag{8-43}$$

Costate equations

$$\dot{\underline{\lambda}} = -W_{ss}\underline{s} - A^T\underline{\lambda} \tag{8-44}$$

Control equations

$$\dot{\underline{U}} = C\underline{U} + D\underline{\lambda} \tag{8-45}$$

8.3.3 Solution for the Initial Costates and Control Time Derivatives

In Eq. (8-43), (8-44), and (8-45), the boundary conditions for $\underline{s}$ are known both initially and finally. Those for $\underline{U}$ are split so that half are known at the initial time and half are known at the final time. Those for $\underline{\lambda}$ are totally unknown. Thus, application of Pontryagin's principle has led to a TPBVP. To obtain the solution for Eq. (8-43), (8-44), and (8-45), the merged state vector is written as

$$\underline{X} = [\underline{s}^T \quad \underline{\lambda}^T \quad \underline{U}^T]^T \tag{8-46}$$

from which it follows that

$$\dot{\underline{X}} = \Omega\underline{X} \tag{8-47}$$

where

$$\Omega = \begin{bmatrix} A & 0 & E \\ -W_{ss} & -A^T & 0 \\ 0 & D & C \end{bmatrix} = \text{constant coefficient matrix}$$

Since Ω is constant, it is well known that Eq. (8-43) possesses the solution

$$\underline{X}(t) = e^{\Omega(t-t_0)} \underline{X}(t_0) \tag{8-48}$$

where $e^{\Omega t}$ is the $(4N + 2kN_c) \times (4N + 2kN_c)$ exponential matrix which is efficiently obtained from a Padé series expansion. Upon setting $e^{\Omega(t-t_0)} = \phi(t,t_0)$ and writing out Eq. (8-48) in partitioned form, one finds

$$\begin{Bmatrix} \underline{s}(t_f) \\ \underline{\lambda}(t_f) \\ \underline{u}_a(t_f) \\ \underline{u}_b(t_f) \end{Bmatrix} = \begin{bmatrix} \phi_{ss} & \phi_{s\lambda} & \phi_{su_a} & \phi_{su_b} \\ \phi_{\lambda s} & \phi_{\lambda\lambda} & \phi_{\lambda u_a} & \phi_{\lambda u_b} \\ \phi_{u_a s} & \phi_{u_a \lambda} & \phi_{u_a u_a} & \phi_{u_a u_b} \\ \phi_{u_b s} & \phi_{u_b \lambda} & \phi_{u_b u_a} & \phi_{u_b u_b} \end{bmatrix} \begin{Bmatrix} \underline{s}(0) \\ \underline{\lambda}(0) \\ \underline{u}_a(0) \\ \underline{u}_b(0) \end{Bmatrix}$$

(8-49)

where

$$\underline{u}_a = [\underline{u}_1^T \quad \underline{u}_2^T \quad \cdots \quad \underline{u}_k^T]^T$$

$$\underline{u}_b = [\underline{u}_{k+1}^T \quad \underline{u}_{k+2}^T \quad \cdots \quad \underline{u}_{2k}^T]^T$$

where $\underline{u}_a$ is known both initially and finally and $\underline{u}_b$ is unknown both initially and finally.

Observing in Eq. (8-49) that the known quantities are $\underline{s}(0)$, $\underline{s}(t_f)$, $\underline{u}_a(0)$, and $\underline{u}_a(t_f)$ and carrying out the partitioned matrix multiplication, solving for $\underline{s}(t_f)$ and $\underline{u}_a(t_f)$, one obtains

$$\begin{Bmatrix} \underline{s}(t_f) \\ \underline{u}_a(t_f) \end{Bmatrix} = \begin{bmatrix} \phi_{ss} & \phi_{su_a} \\ \phi_{u_a s} & \phi_{u_a u_a} \end{bmatrix} \begin{Bmatrix} \underline{s}(0) \\ \underline{u}_a(0) \end{Bmatrix} + \begin{bmatrix} \phi_{s\lambda} & \phi_{su_b} \\ \phi_{u_a \lambda} & \phi_{u_a u_b} \end{bmatrix} \begin{Bmatrix} \underline{\lambda}(0) \\ \underline{u}_b(0) \end{Bmatrix} \tag{8-50}$$

Setting $\underline{u}_a(0) = \underline{u}_a(t_f) = \underline{0}$ (i.e., turning the control system off initially and finally) in Eq. (8-50) (see Eq. (8-28) and (8-29)) and solving Eq. (8-50) for $\underline{\lambda}(0)$ and $\underline{u}_b(0)$, one finds

$$\begin{bmatrix} \phi_{s\lambda} & \phi_{su_b} \\ \phi_{u_a \lambda} & \phi_{u_a u_b} \end{bmatrix} \begin{Bmatrix} \underline{\lambda}(0) \\ \underline{u}_b(0) \end{Bmatrix} = \begin{Bmatrix} \underline{s}_f - \phi_{ss}\underline{s}(0) \\ - \phi_{u_a s}\,\underline{s}(0) \end{Bmatrix} \tag{8-51}$$

Equation (8-51) is in the standard form for the linear algebraic equation $A\underline{x} = \underline{b}$, which can be solved via Gaussian elimination for $\underline{x}$, to yield $\underline{\lambda}(0)$ and $\underline{\dot{u}}_b(0)$. The optimal control time histories are obtained by integrating Eq. (8-43), (8-44), and (8-45) subject to the given state and control boundary conditions $\underline{s}(0)$ and $\underline{u}_a(0)$ and the costate and control rate boundary conditions provided by Eq. (8-51). Cases 2 and 3 in Section 8.6 present example maneuvers using the formulations of this section.

8.3.4 Free Final Angle Maneuvers with Control-Rate Penalties Included

Since the analytical developments for solving the problem of free final angle maneuvers are similar to those shown in Section 8.2, only key equations are given.

As shown in Section 8.2, the boundary condition for the free final angle corresponds to setting the final value of the costate for the rigid-body angle to zero (see Eq. (8-15) and (8-16)). As a result, the set of prescribed boundary conditions follow as

$$\underline{\zeta}_0 = [\theta_0 \; \underline{n}^T(t_0)]^T \qquad \dot{\underline{\zeta}}_0 = [\dot{\theta}_0 \; \dot{\underline{n}}^T(t_0)]^T \qquad \underline{u}_{a_0} = \underline{u}_a(t_0) \tag{8-52}$$

$$\underline{n}_f = \underline{n}(t_f) \qquad \dot{\underline{\zeta}}_f = [\dot{\theta}_f \; \dot{\underline{n}}^T(t_f)]^T \qquad \underline{u}_{a_f} = \underline{u}_a(t_f) \tag{8-53}$$

and

$$\Lambda_1(t_f) = 0 \tag{8-54}$$

where we impose the requirements that $\underline{n}(t_f) = \dot{\underline{n}}(t_f) = \underline{0}$ in Eq. (8-53) and $\underline{u}_a(t_0) = \underline{u}_a(t_f) = \underline{0}$ in Eq. (8-52) and (8-53).

Using the performance index of Eq. (8-31) and formulating the solution in physical space, the state, costate, and control differential equations follow as:

<u>State equation</u>

$$\dot{\underline{\Sigma}} = A\underline{\Sigma} + E\underline{\Lambda} \tag{8-55}$$

<u>Costate equation</u>

$$\dot{\underline{\Lambda}} = -W_{ss}\underline{\Sigma} - A^T\underline{\Lambda} \tag{8-56}$$

<u>Control equation</u>

$$\dot{\underline{U}} = C\underline{U} + D\underline{\Lambda} \tag{8-57}$$

where

$$\underline{\Sigma} = [\underline{\zeta}^T \quad \underline{\dot{\zeta}}^T]^T$$

Writing the merged state vector as

$$\underline{X} = [\underline{\Sigma}^T \quad \underline{\Lambda}^T \quad \underline{U}^T]^T \tag{8-58}$$

the solution for Eq. (8-55), (8-56), and (8-57) can be written as

$$\underline{X}(t) = e^{\Omega(t-t_0)}\underline{X}(t_0) \tag{8-59}$$

where Ω is defined in Eq. (8-47). Setting $e^{\Omega(t-t_0)} = \Phi(t,t_0)$ and performing the partitioned matrix multiplication, solving for $\Sigma_2(t_f),\ldots,\Sigma_{2N}(t_f)$, $\Lambda_1(t_f)$ in Eq. (8-59), the linear equation to be solved for $\underline{\Lambda}(0)$ and $\underline{u}_b(0)$ follows as

$$[A]\underline{a} = \underline{b} - [B]\underline{c} \tag{8-60}$$

where

$$[A] = \begin{bmatrix} \Phi_{\Sigma_2\Lambda_1} & \cdots & \Phi_{\Sigma_2\Lambda_{2N}} & \Phi_{\Sigma_2 u_{b_1}} & \cdots & \Phi_{\Sigma_2 u_{b_\nu}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{\Sigma_{2N}\Lambda_1} & \cdots & \Phi_{\Sigma_{2N}\Lambda_{2N}} & \Phi_{\Sigma_{2N} u_{b_1}} & \cdots & \Phi_{\Sigma_{2N} u_{b_\nu}} \\ \Phi_{\Lambda_1\Lambda_1} & \cdots & \Phi_{\Lambda_1\Lambda_{2N}} & \Phi_{\Lambda_1 u_{b_1}} & \cdots & \Phi_{\Lambda_1 u_{b_\nu}} \\ \Phi_{u_{a_1}\Lambda_1} & \cdots & \Phi_{u_{a_1}\Lambda_{2N}} & \Phi_{u_{a_1} u_{b_1}} & \cdots & \Phi_{u_{a_1} u_{b_\nu}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{u_{a_\nu}\Lambda_1} & \cdots & \Phi_{u_{a_\nu}\Lambda_{2N}} & \Phi_{u_{a_\nu} u_{b_1}} & \cdots & \Phi_{u_{a_\nu} u_{b_\nu}} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \Phi_{\Sigma_2 \Sigma_1} & \cdots & \Phi_{\Sigma_2 \Sigma_{2N}} \\ \vdots & & \vdots \\ \Phi_{\Sigma_{2N} \Sigma_1} & \cdots & \Phi_{\Sigma_{2N} \Sigma_{2N}} \\ \Phi_{\Lambda_1 \Sigma_1} & \cdots & \Phi_{\Lambda_1 \Sigma_{2N}} \\ \Phi_{u_{a_1} \Sigma_1} & \cdots & \Phi_{u_{a_1} \Sigma_{2N}} \\ \vdots & & \vdots \\ \Phi_{u_{a_\nu} \Sigma_1} & \cdots & \Phi_{u_{a_\nu} \Sigma_{2N}} \end{bmatrix}$$

$$\underline{a} = [\Lambda_1(0) \ldots \Lambda_{2N-1}(0) \quad \Lambda_{2N}(0) \quad u_{b_1}(0) \ldots u_{b_\nu}(0)]^T$$

$$\underline{b} = [\Sigma_2(t_f) \ldots \Sigma_{2N}(t_f) \quad \Lambda_1(t_f) \quad u_{a_1}(t_f) \ldots u_{a_\nu}(t_f)]^T$$

$$\underline{c} = [\Sigma_1(0) \ldots \Sigma_{2N-1}(0) \quad \Sigma_{2N}(0) \quad u_{a_1}(0) \ldots u_{a_\nu}(0)]^T$$

$$\nu = kNc$$

Eq. (8-60) is in the standard form for the linear algebraic equation $A\underline{x} = \underline{b}$, which can be solved via Gaussian elimination for $\underline{x}$, yielding $\underline{\Lambda}(0)$ and $\underline{u}_b(0)$.

The optimal control time histories are obtained upon integrating Eq. (8-55), (8-56), and (8-57) subject to $\underline{\Sigma}(0)$, $\underline{u}_a(0)$ given by Eq. (8-52) and $\underline{\Lambda}(0)$, $\underline{u}_b(0)$ given by Eq. (8-60).

Eq. (8-60) provides the initial costate and control rates in terms of physical space coordinates. However, if the maneuver simulations are modeled in terms of modal space coordinates, the following two transformations permit the use of the results of this section. First, the modal space state transition matrix, $\phi(t_f,0)$, is mapped to physical space via the transformation

$$\Phi(t_f,0) = \Theta\phi(t_f,0)\Theta^{-1} \tag{8-61}$$

where

$$\Theta = \begin{bmatrix} E & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & ME & 0 & 0 & 0 \\ 0 & 0 & 0 & ME & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$\Theta^{-1} = \begin{bmatrix} E^TM & 0 & 0 & 0 & 0 & 0 \\ 0 & E^TM & 0 & 0 & 0 & 0 \\ 0 & 0 & E^T & 0 & 0 & 0 \\ 0 & 0 & 0 & E^T & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

Second, introducing the required partitions of Eq. (8-61) into Eq. (8-60) and solving for $\underline{\Lambda}(0)$ and $\underline{u}_b(0)$, the modal space initial conditions are obtained from the transformation

$$\begin{Bmatrix} \underline{s}(0) \\ \underline{\lambda}(0) \\ \underline{u}_a(0) \\ \underline{u}_b(0) \end{Bmatrix} = \Theta^{-1} \begin{Bmatrix} \underline{\Sigma}(0) \\ \underline{\Lambda}(0) \\ \underline{u}_a(0) \\ \underline{u}_b(0) \end{Bmatrix} \tag{8-62}$$

The modal space optimal control time histories are obtained upon integrating the modal space form of Eq. (8-55), (8-56), and (8-57) subject to the initial conditions provided by Eq. (8-62). Case 4 in Section 8.6 provides an example maneuver using the formulations of this section.

8.3.5 Selection Technique for Weighting Matrices W_{ss} and W_{ii} in the Control-Rate Penalty Formulation

It has been found that the particular choice of weighting matrices significantly affects the computational effort and resulting maneuvers. A brief study of the effects has been performed.

In the example maneuvers of Section 8.6, the following block diagonal forms of the weighting matrices have been used

$$W_{ss} = w_{ss} \begin{bmatrix} 10^{-2} & 0 \\ 0 & I \end{bmatrix} \quad (2N \times 2N)$$

$$W_{ii} = w_{ii} \begin{bmatrix} 10^{-\rho} & 0 \\ 0 & I \end{bmatrix} \quad (N_c \times N_c)$$

where w_{ss}, w_{ii} are scalar quantities and ρ is defined in the text that follows.

Since the problem being solved is a slewing maneuver, the rigid body angular displacement is not penalized. Hence, the first element of W_{ss} is set two orders of magnitude less than the other diagonal elements. The first element of W_{ss} is not set to zero because this causes repeated zero eigenvalues in Ω of Eq. (8-47), which can potentially lead to numerical diffuclties in calculating the state transition matrix.

The value of ρ in W_{ii} determines the participation of each set of independent controllers in the slewing maneuver. Since it is desirable to have the rigid-body torque execute most of the slewing maneuver while the appendage torques function mainly as vibration controllers, ρ is set to 2 or 3 to provide a smaller penalty on the rigid-body torque.

In a preliminary parametric study, it has been found that by adjusting the norms of the weighting matrices W_{ss}, W_{00}, and W_{ii} (for $i = 1,\ldots,k-1$) to several orders of magnitude smaller than the norm of W_{kk}, the eigenvalue bandwidth of Ω in Eq. (8-47) can be reduced significantly; thus, the sensitivity of the optimal control problem decreases. However, the decrease in eigenvalue bandwidth of Ω is obtained at the expense of increased peak flexural deformations during the maneuver, though the time histories of the modal amplitudes are considerably smoother. An assessment of the impact this weighting matrix scheme has on the unmodeled degrees of freedom is a topic of further interest.

8.4 Single Axis Retargeting Maneuvers for a Rigid Spacecraft

The problem addressed in this section is a single-axis maneuver of a rigid spacecraft which is slewing to engage a moving target. The retargeting maneuver is complicated by the fact that the following parameters and boundary conditions are unknown:

(1) The maneuver time, $(t_f - t_0)$.

(2) The final engagement angle, θ_f.

(3) The final engagement angular rate, $\dot{\theta}_f$.

(4) The final control torque, $\underline{u}_f$.

(5) The final control torque rate, $\dot{\underline{u}}_f$.

The performance index selected for the optimal control problem represents a tradeoff between elapsed maneuver time, state, and control-rate penalties. A rigid spacecraft model is used instead of a flexible vehicle to simplify the calculations and provide useful insight for solving the flexible body case. In the developments that follow, the target trajectory is assumed to be known. Since work on this topic is preliminary, only the solution strategy is presented.

The solution procedure consists of the following three steps. First, the linearized free final time problem is solved. Second, the nonlinear solution is obtained using the final time and initial costates from the first step as initial iteratives. Third, a grid search is performed in time to find the final time that minimizes the performance index for the nonlinear problem.

The target trajectory is assumed to be

$$\theta(t,t_0) = \tan^{-1} \frac{(y_0 + V(t - t_0))}{x_0} \qquad (8\text{-}63)$$

which represents a linear flyby target motion, where x_0, y_0, and $\dot{V}$ are constants that correspond to the target's initial position and velocity (see Figure 8-3).

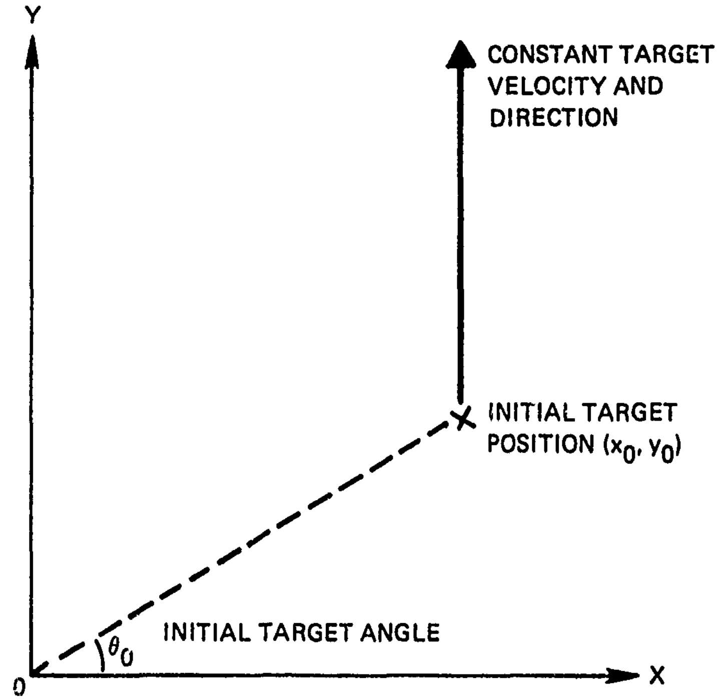


Figure 8-3. Model of a linear flyby target motion.

8.4.1 Solution for the Linearized Free Final Time Problem

The solution of the linearized problem provides estimates of the optimal final time, initial costates, and control time derivatives for the nonlinear solution.

The state space form of the equation of motion for the linear problem is given by

$$\begin{Bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{Bmatrix} = \begin{Bmatrix} s_2 \\ u/I \end{Bmatrix} \tag{8-64}$$

where s_1 is the angular displacement, s_2 is the angular velocity, u is the control torque, and I is the moment of inertia of the vehicle. The optimal control problem is to seek the solution to Eq. (8-64) that

(1) Minimizes the performance index

$$J = \int_0^{t_f} [\alpha + \frac{1}{2} W_{00} u^2 + \frac{1}{2} W_{11} \dot{u}^2 + \frac{1}{2} W_{22} \ddot{u}^2] \, dt \tag{8-65}$$

where α is a weight penalizing elapsed time, and W_{00}, W_{11}, W_{22} are scalar weights on the control, first, and second control time derivatives, respectively.

(2) Satisifies the terminal constraints

$$s_1(t_0) = s_{1_0} \qquad s_2(t_0) = s_{2_0} \tag{8-66}$$

$$u(t_0) = 0 \qquad \dot{u}(t_0) = 0 \tag{8-67}$$

$$s_1(t_f) = \theta(t_f, t_0) \qquad s_2(t_f) = \dot{\theta}(t_f, t_0) \tag{8-68}$$

$$u(t_f) = I\ddot{\theta}(t_f, t_0) \qquad \dot{u}(t_f) = I\dddot{\theta}(t_f, t_0) \tag{8-69}$$

where

$$t_f = \text{free} \tag{8-70}$$

and $\theta(t, t_0)$ is the angular target motion given by Eq. (8-63).

The final boundary conditions given by Eq. (8-69) are imposed in order to establish a terminal torque and torque rate that will allow accurate pointing and tracking of the target at the end of the maneuver. The target motion in terms of $\theta(t,t_0)$, $\dot{\theta}(t,t_0)$, $\ddot{\theta}(t,t_0)$, and $\dddot{\theta}(t,t_0)$ is assumed to be available from some estimation process, for practical applications.

Application of Pontryagin's principle leads to the following necessary conditions

$$\dot{s}_1 = s_2 \qquad (8\text{-}71)$$

$$\dot{s}_2 = u/I \qquad (8\text{-}72)$$

$$\dot{\lambda}_1 = 0 \qquad (8\text{-}73)$$

$$\dot{\lambda}_2 = -\lambda_1 \qquad (8\text{-}74)$$

$$W_{00}u - W_{11}\ddot{u} + W_{22}\dddot{u} = \frac{-\lambda_2}{I} \qquad (8\text{-}75)$$

On imposing the terminal constraints listed in Eq. (8-68) and (8-69), the transversality conditions governing the optimal maneuver follow as

$$G_1 \equiv -\lambda_1(t_f)\dot{\theta}(t_f,t_0) - \lambda_2(t_f)\ddot{\theta}(t_f,t_0) + \lambda_1(t_f)s_2(t_f) + \lambda_2(t_f)u(t_f)/I + \alpha$$

$$+ \frac{1}{2}W_{00}u^2(t_f) + \frac{1}{2}W_{11}\dot{u}^2(t_f) + \frac{1}{2}W_{22}\ddot{u}^2(t_f) = 0 \qquad (8\text{-}76)$$

$$G_2 \equiv s_1(t_f) - \theta(t_f,t_0) = 0 \qquad (8\text{-}77)$$

$$G_3 \equiv s_2(t_f) - \dot{\theta}(t_f,t_0) = 0 \qquad (8\text{-}78)$$

$$G_4 \equiv u(t_f) - I\ddot{\theta}(t_f,t_0) = 0 \qquad (8\text{-}79)$$

$$G_5 \equiv \dot{u}(t_f) - I\dddot{\theta}(t_f,t_0) = 0 \qquad (8\text{-}80)$$

To cast Eq. (8-75) into state space form, the control state is defined as

$$[u_1 \quad u_2 \quad u_3 \quad u_4] = [u \quad \dot{u} \quad \ddot{u} \quad \dddot{u}] \qquad (8\text{-}81)$$

Subject to Eq. (8-81), Eq. (8-71) to (8-75) can be written in the merged state vector form

$$\underline{\dot{x}} = A\underline{x} \tag{8-82}$$

where

$$\underline{x} = [s_1 \quad s_2 \quad \lambda_1 \quad \lambda_2 \quad u_1 \quad u_2 \quad u_3 \quad u_4]^T$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{IW_{22}} & -\frac{W_{00}}{W_{22}} & 0 & \frac{W_{11}}{W_{22}} & 0 \end{bmatrix}$$

Since A is constant, it is well known that Eq. (8-82) possesses the solution

$$\underline{x}(t) = e^{A(t-t_0)}\underline{x}(t_0) \tag{8-83}$$

where

$e^{A(t_f-t_0)}$ is the exponential matrix

Setting

$$e^{A(t_f-t_0)} = \phi(t_f,t_0)$$

introducing the appropriate partitions of $\phi(t_f, t_0)$ into Eq. (8-50) and rearranging, the linear system defining the solution for $\lambda_1(t_0)$, $\lambda_2(t_0)$, $u_3(t_0)$, and $u_4(t_0)$ follows as

$$\begin{bmatrix} \phi_{s\lambda} & \phi_{su_b} \\ \phi_{u_a\lambda} & \phi_{u_a u_b} \end{bmatrix} \begin{Bmatrix} \underline{\lambda}(0) \\ \underline{u}_b(0) \end{Bmatrix} = \begin{Bmatrix} \underline{s}(t_f) - \phi_{ss}\underline{s}(0) \\ \underline{u}_a(t_f) - \phi_{u_a s}\underline{s}(0) \end{Bmatrix} \tag{8-84}$$

where

$$\underline{s} = [s_1 \quad s_2]^T$$

$$\underline{\lambda} = [\lambda_1 \quad \lambda_2]^T$$

$$\underline{u}_a = [u_1 \quad u_2]^T$$

$$\underline{u}_b = [u_3 \quad u_4]^T$$

The optimal control time histories follow on integrating Eq. (8-82) subject to $\underline{s}(0)$ and $\underline{u}_a(0)$ given by Eq. (8-66) and (8-67) and $\underline{\lambda}(0)$ and $\underline{u}_b(0)$ given by Eq. (8-84).

Due to the nonlinear character of the transversality conditions, the necessary equations are solved by iteration. Starting iteratives for the transversality conditions are obtained by assuming that the final time is known. As a result, a number of final time grid points are selected. At each final time grid point, the optimal control solution is obtained subject to satisfying the transversality conditions of Eq. (8-77) through (8-80), and the performance index is computed. Once a minimum is found for the performance index within three consecutive final time grid points, the performance index is modeled by $J_k(t_f^k) = C_0 + C_1 t_f^k + C_2(t_f^k)^2$ where J_k denotes the k^{th} performance index and t_f^k denotes the k^{th} final time grid point. Upon solving for C_0, C_1, and C_2, the starting iterative for t_f is obtained by setting the derivative of $J_k(t)$ equal to zero, yielding $t_f = -C_1/2C_2$.

Integrating Eq. (8-82) and substituting the final states, costates, and control states into Eq. (8-76) to (8-80) provides the errors in satisfying the necessary conditions. The corrections to the initial costates, control time derivatives, and final time are obtained from the following equation

$$-[g_{ij}] \begin{Bmatrix} \Delta C_1 \\ \Delta C_2 \\ \Delta C_3 \\ \Delta C_4 \\ \Delta C_5 \end{Bmatrix} = \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{Bmatrix} \tag{8-85}$$

where

$$[C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5] = [\lambda_1(t_0) \quad \lambda_2(t_0) \quad u_3(t_0) \quad u_4(t_0) \quad t_f]$$

$$g_{ij} = \frac{\partial G_i}{\partial C_j} \qquad i,j = 1,\ldots,5$$

and the desired corrections are denoted by

$$\Delta C_i \text{ for } i = 1,\ldots,5$$

Using the updated final time, the terminal errors for Eq. (8-76) through (8-80) are computed on integrating Eq. (8-82), and if necessary, the process is repeated until the final time converges.

8.4.2 Retargeting Maneuvers with Nonlinear System Dynamics

A retargeting maneuver is considered where the system dynamics are nonlinear (this example is similar to the nonlinear flexible problem shown in Reference 8-1) and time-varying weights are included in the performance index. Specifically, the equation of motion is assumed to be given by

$$I\dot{s}_2 = u - \kappa s_2^2 \tag{8-86}$$

and the new performance index follows as

$$J = \int_0^{t_f} [\alpha + \frac{1}{2} W_{00}u^2 + \frac{1}{2} W_{11}\dot{u}^2 + \frac{1}{2} W_{22}\ddot{u}^2 + \frac{1}{2} \gamma(t)(s_1 - \theta)^2$$

$$+ \frac{1}{2} \beta(t)(s_2 - \dot{\theta})^2] \, dt \qquad (8\text{-}87)$$

where

$$\gamma(t) = C_1(e^{\Gamma t} - 1), \ \beta(t) = C_2(e^{\alpha t} - 1)$$

and C_1, C_2, Γ, and α are constants.

The performance index now includes time-varying penalties on the differences in angular displacement and angular velocity between the spacecraft and the target.

The problem is solved as a fixed final time maneuver using the optimal final time found in the free final time problem in Section 8.4.1. The nonlinear solution is obtained by introducing a relaxation process (see Reference 8-1, Section 8.6), where the nonlinearities are introduced slowly into the linear solution obtained in Section 8.4.1.

Application of Pontryagin's principle now results in the following necessary conditions

$$\dot{s}_1 = s_2 \qquad (8\text{-}88)$$

$$\dot{s}_2 = \frac{u}{I} - \frac{K}{I} s_2^2 \qquad (8\text{-}89)$$

$$\dot{\lambda}_1 = -\gamma(t)(s_1 - \theta) \qquad (8\text{-}90)$$

$$\dot{\lambda}_2 = -\beta(t)(s_2 - \dot{\theta}) - \lambda_1 - 2\lambda_2 \left(\frac{K}{I}\right)s_2 \qquad (8\text{-}91)$$

$$W_{00} + \frac{\lambda_2}{I} - W_{11}\ddot{u} + W_{22}\dddot{u} = 0 \qquad (8\text{-}92)$$

The terminal boundary conditions listed in Eq. (8-66) through (8-69) are assumed for this problem, and the initial estimates for $\underline{\lambda}(0)$ and $\underline{u}_b(0)$ are given by Eq. (8-84). Written in merged state vector form, Eq. (8-88) through (8-92) become

$$\dot{\underline{x}} = A\underline{x} + \sigma_k \underline{b} \tag{8-93}$$

where

$$\underline{b} = \begin{Bmatrix} 0 \\ -\kappa s_2^2/I \\ -\gamma(t)(s_1 - \theta) \\ -\beta(t)(s_2 - \dot{\theta}) - 2\lambda_2\left(\frac{\kappa}{I}\right)s_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

and σ_k is the k^{th} relaxation parameter, where the sequence of parameters $\{\sigma_0 = 0 < \sigma_1 < \ldots < \sigma_{r-1} < \sigma_r = 1\}$ has been introduced and r is preset. The solution for Eq. (8-93) is refined iteratively using the following equation

$$\left[\frac{\partial \underline{x}_f^T}{\partial \underline{x}_0}\right]^T \Delta\underline{x}_0 = \underline{x}_{fD} - \underline{x}_{fI} \tag{8-94}$$

where

$\underline{x}_{fD}$ is the desired final state vector

$\underline{x}_{fI}$ is the integrated final state vector

$[\partial \underline{x}_f^T/\partial \underline{x}_0]^T$ is the state transition matrix

$\Delta\underline{x}_0$ is the correction vector

Assuming that $[\partial \underline{x}_f^T / \partial \underline{x}_0]^T$ in Eq. (8-94) can be expressed in the form

$$\left[\frac{\partial \underline{x}_f^T}{\partial \underline{x}_0}\right]^T = \phi(t_f,t_0)Q(t_f,t_0) \tag{8-95}$$

where $\phi(t_f,t_0)$ is the state transition matrix for the linear time-invariant part of the problem, and $Q(t_f,t_0)$ is the state transition matrix for the nonlinear part of the problem, it can be shown using Lagrange's variation of parameters method that $Q(t_f,t_0)$ satisfies the equation

$$Q(t_f,t_0) = I + \sigma_k \int_{t_0}^{t_f} \phi^{-1}(\tau,t_0)\left[\frac{\partial \underline{b}^T(\tau)}{\partial \underline{x}}\right]^T \phi(\tau,t_0)Q(\tau,t_0)\, d\tau \tag{8-96}$$

where the relaxation parameter, σ_k, has been introduced artificially. Observing that Eq. (8-96) is an integral equation for $Q(t,t_0)$, and using the method of successive substitutions, one finds the following equation

$$Q(t_f,t_0) \approx I + \sigma_k \int_{t_0}^{t_f} \phi^{-1}(\tau,t_0)\left[\frac{\partial \underline{b}^T(\tau)}{\partial \underline{x}}\right]^T \phi(\tau,t_0)\, d\tau \tag{8-97}$$

where the higher order terms in σ_k have been assumed to be small.

The solution procedure is summarized as follows. First, the relaxation parameters σ_k are chosen such that $\{\sigma_0 = 0 < \sigma_1 < \sigma_2 \ldots < \sigma_r = 1\}$. Second, the initial costates and control time derivatives for the linear solution are used as initial conditions for the integration of Eq. (8-93) with k = 1 and Eq. (8-97). The final states and controls are compared with the desired values at the final time, and if the terminal errors exceed a prescribed tolerance, then corrections to the initial costates and control time derivatives are calculated from Eq.(8-94), (8-95), and (8-97). The integrations and corrections are repeated until the norm of the terminal error vector is sufficiently small. The index on the relaxation parameter is increased, and the iteration process continues until the solution for $\sigma_r = 1$ is found. The solution for $\sigma_r = 1$ yields the desired nonlinear solution. It is anticipated that Eq. (8-97) will lead to significant computational savings when $[\partial \underline{b}^T(\tau)/\partial \underline{x}]$ is sparse.

The optimal final time for the nonlinear solution is obtained by a grid search for the final time that minimizes the performance index of Eq. (8-87). That is, a series of nonlinear solutions and performance indices are calculated for different final times. A quadratic fit is then performed to estimate the final time that minimizes the performance index.

8.5 Preliminary Study of Slewing Maneuvers for the ACOSS Model 2

Several single-axis slewing maneuvers have been computed for a modified version of the ACOSS Model 2. The modifications, which simplify the calculations, include:

(1) Locked isolator springs.

(2) A symmetric mass distribution for the vehicle about its center of mass.

The "rigid-body" torque is applied in the z direction at the center of the equipment section (node 44). The formulation of Section 8.3 is used with k = 2 in Eq. (8-30) for the performance index. The mass, stiffness, and actuation matrices are computed by the Dynamic Interaction Simulation of Controls and Structures program (DISCOS) which uses NASTRAN finite-element data.

The solution procedure is as follows. A number of modes to be controlled is chosen. The optimal control time history is computed, using the formulation of Section 8.3 and the mass, stiffness, and actuation matrices from DISCOS. The control time history is then input to DISCOS which simulates the maneuver and treats all nonlinear kinematic effects.

Results from the DISCOS simulations show that the uncontrolled symmetric modes have small vibrations. Though the controlled antisymmetric modes have larger peak amplitudes, the results show that the modal amplitudes are zero at and beyond the final time of the maneuvers, which is the expected response. An example maneuver is shown in Section 8.6, Case 5.

8.6 Example Maneuvers

This section presents example maneuvers for the formulations discussed in Sections 8.2 through 8.5. For Cases 1 to 4, the geometry of Figure 8-1 is assumed with the following configuration parameters: the moment of inertia of the undeformed structure, I, is 7000 kg-m^2; the mass per unit length of the four identical elastic appendages, ρ, is 0.004 kg/m; the length of each cantilevered appendage, L, is 150 m; the flexural rigidity of the appendages, EI, is 1500 kg-m^3/s^2; and the radius of the rigid hub, r, is 1 m. In the integrations over the mass stiffness distributions, the radius of the hub is not neglected in comparison to the appendage length. The following comparison functions are used as assumed modes

$$\phi_p(x - r) = 1 - \cos \frac{p\pi(x - r)}{L} + \frac{1}{2} (-1)^{p+1} \left(\frac{p\pi(x - r)}{L}\right)^2$$

$$p = 1,2,\ldots,\infty$$

which satisfy the geometric and physical boundary conditions

$$\phi_p\Big|_{x=r} = \phi'_p\Big|_{x=r} = \phi''_p\Big|_{x=r+L} = \phi'''_p\Big|_{x=r+1} = 0$$

of a clamped-free appendage. For Case 5, the ACOSS Model 2 is used where the isolation springs are locked and the mass distribution has been modified.

In Cases 1 to 4, all the W_{ss} matrices are set to diagonal matrices in physical space, and then mapped into modal space using

$$[W_{ss}]_{modal} = \begin{bmatrix} E^T & 0 \\ 0 & E^T \end{bmatrix} [W_{ss}]_{physical} \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

For Case 5, the W_{ss} matrix is set to a diagonal matrix in modal space. Referring to Table 8-1 and Figures 8-4 to 8-14, the example maneuvers are described as follows.

8.6.1 Case 1 (Figure 8-4)

Case 1 is a 60-second free final angle rotation reversal maneuver using the formulation of Section 8.2. The structure returns to its original angular position at the end of the maneuver. The modal amplitudes are negative and the control torques are positive throughout the maneuver. In addition, the control torques have jump discontinuities at the initial and final time.

8.6.2 Cases 2 and 3 (Figures 8-5 through 8-7)

Cases 2 and 3 are 60-second rest-to-rest maneuvers using the formulation of Section 8.3. The states, controls, and control rates are included in the performance index for Case 2, while the second and third time derivatives of the controls are included additionally for Case 3. For both cases, the control torques at the initial and final times are all zero; therefore, they are continuous at those points. The slope of the maneuver angle versus time plots are very small at the initial and final times which is consistent with a smooth modal amplitude history. Case 3 differs from Case 2 in that there is a 30-to-1 reduction in the first mode peak amplitude, a 2-to-1 reduction in the second mode peak amplitude, and a 20-to-1 reduction in the peak tip deflection. On the other hand, Case 3 shows a 1-to-1.3 increase in

Table 8-1. Parameters used in test case maneuvers.

Case No.	No. of Modes Controlled	No. of Controls	θ_0 (rad)	$\dot{\theta}_0$ (rad)	θ_f (rad)	$\dot{\theta}_f$ (rad)	W_{ss}	W_{00}	W_{11}	W_{22}	W_{33}
1	2	5	0	-0.5	free	0.5	$10^{-2}\hat{I}$	$\tilde{I}$	---	---	---
2	2	5	0	0	π	0	$10^{-3}\hat{I}$	$10^{-9}I$	$\tilde{I}$	---	---
3	2	5	0	0	π	0	$10^{-3}\hat{I}$	$10^{-9}I$	$10^{-9}I$	$10^{-9}I$	$\tilde{I}$
4	2	5	0	-0.5	free	0.5	$10^{-3}\hat{I}$	$10^{-9}I$	$10^{-9}I$	$\tilde{I}$	---
5	2	1	0	0	$\pi/36$	0	$10^{-7}\hat{I}$	$10^{-4}I$	$10^{-4}I$	I	---

$\hat{I}$ is an identity matrix with the first element set to 10^{-2}, setting a lower weight on the maneuver angle.

$\tilde{I}$ is an identity matrix with the first element set to 10^{-3}, setting a lower weight on the rigid-body control or control time derivative.

the rigid-body peak torque requirement and a 1-to-8.4 increase in the elastic appendage peak torque requirement when compared with Case 2. The slopes of the control time history plots for Case 3 are zero at the initial and final times because of the higher order time derivatives in the performance index.

8.6.3 Case 4 (Figures 8-8 and 8-9)

Case 4 is a 60-second, free final angle, rotation reversal maneuver using the formulation of Section 8.3.4 with the performance index penalizing the states, controls, and the first and second time derivatives of the controls. Compared with Case 1, Case 4 shows a 19-to-1 reduction in the first mode peak amplitude and a 2-to-1 reduction in the second mode peak amplitude. The rigid-body peak torque, on the other hand, increases by a 1-to-1.3 ratio and the appendage peak torque by a 1-to-13 ratio. The initial and final control torques are zero, and therefore continuous, while in Case 1 they are discontinuous at the initial and final times. The modal amplitude and control torque histories are much smoother in Case 4 than in Case 1. From these results, it can be concluded that the inclusion of control-rate penalties in the performance index produces torque and modal amplitude time histories which are much smoother than those produced by the formulation without the control-rate penalty.

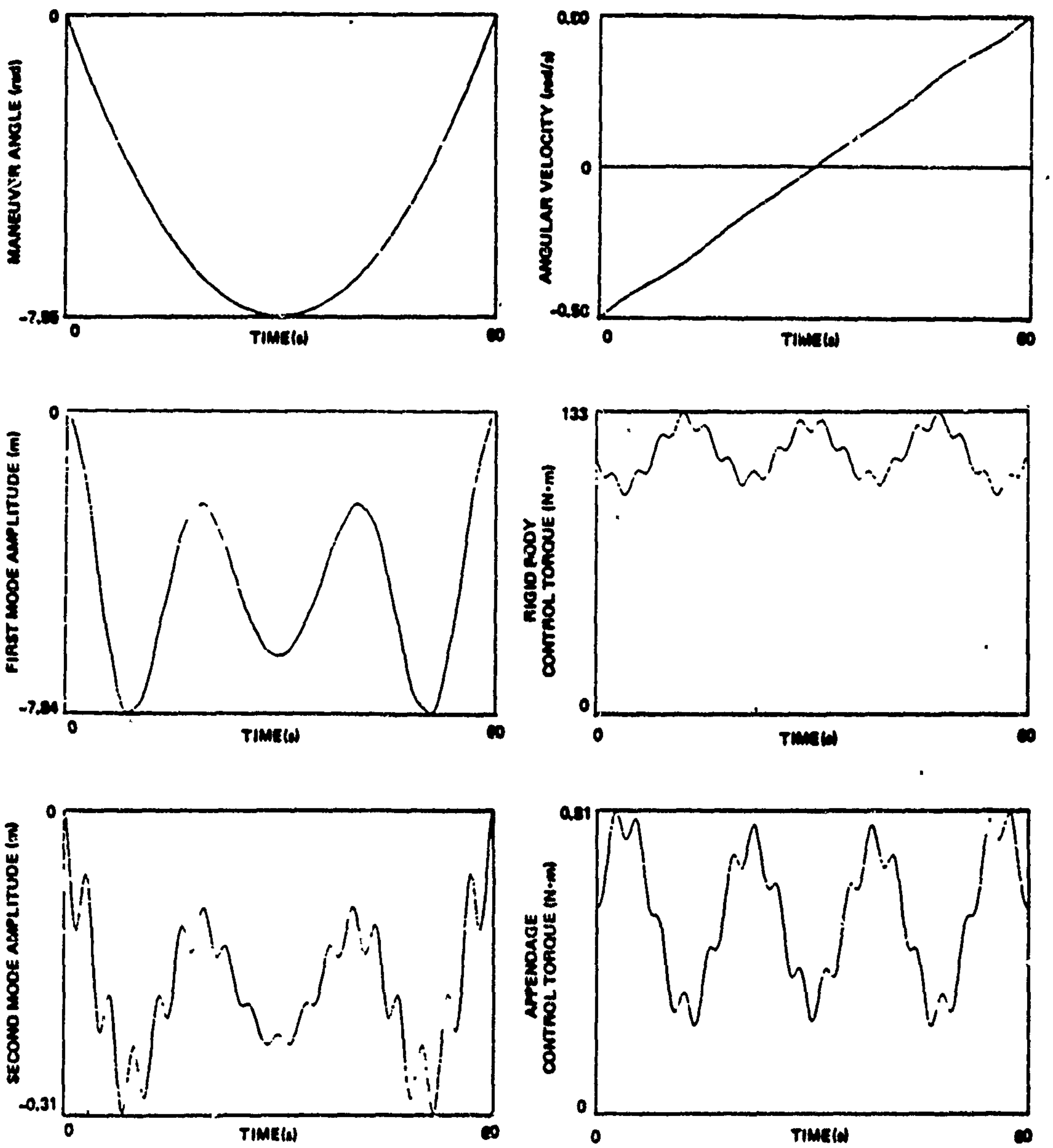


Figure 8-4. Case 1, free final angle for rotation reversal maneuver.

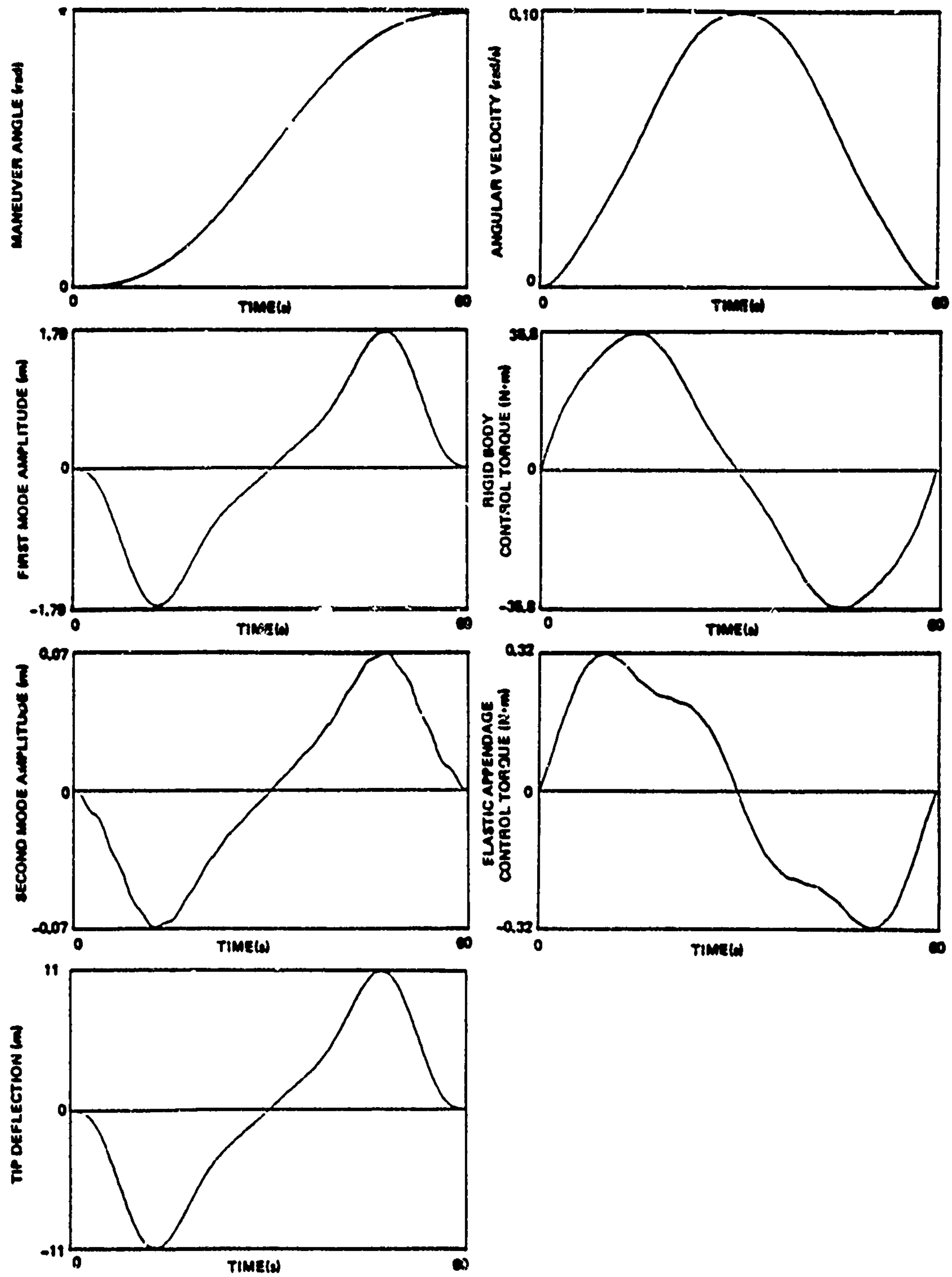


Figure 8-5. Case 2, performance index including penalty on control rate.

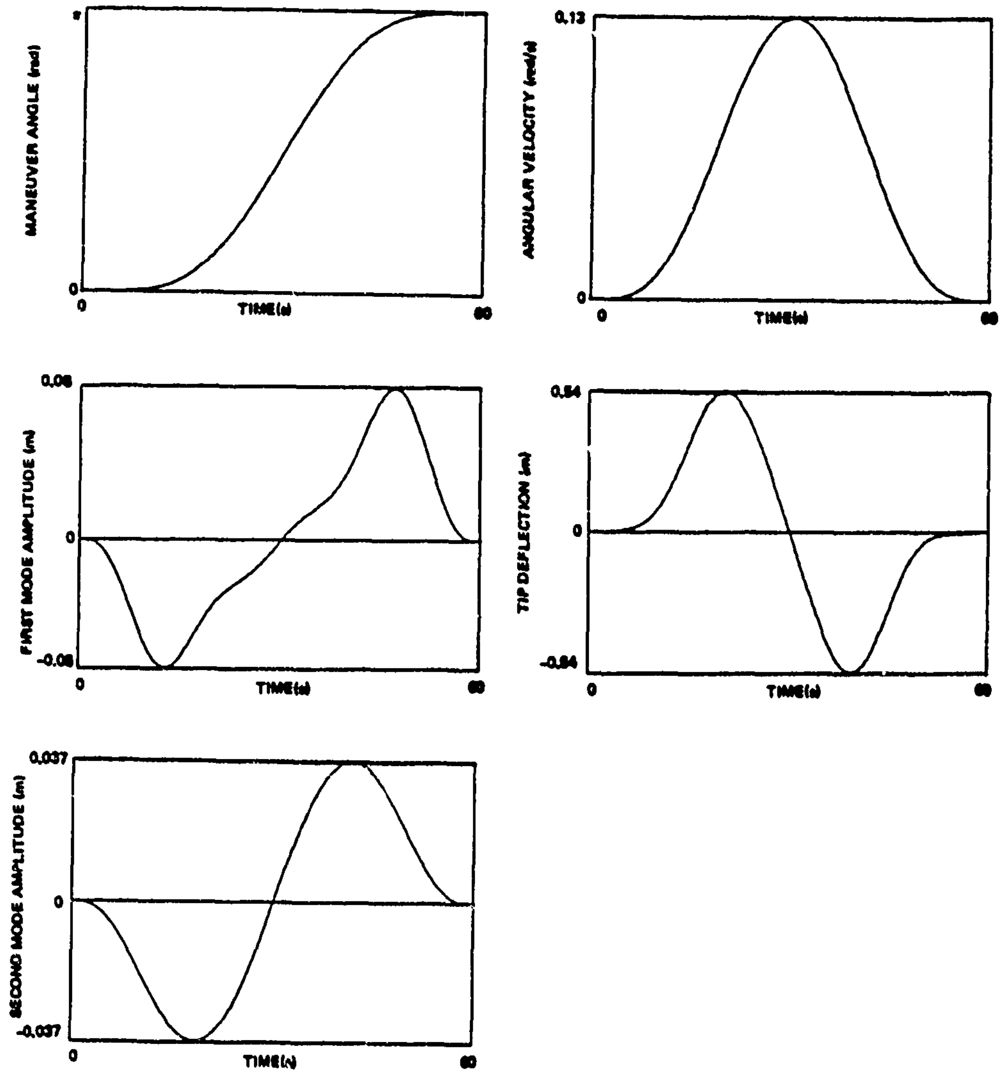


Figure 8-6. Case 3, performance index including higher-order time derivatives of the controls.

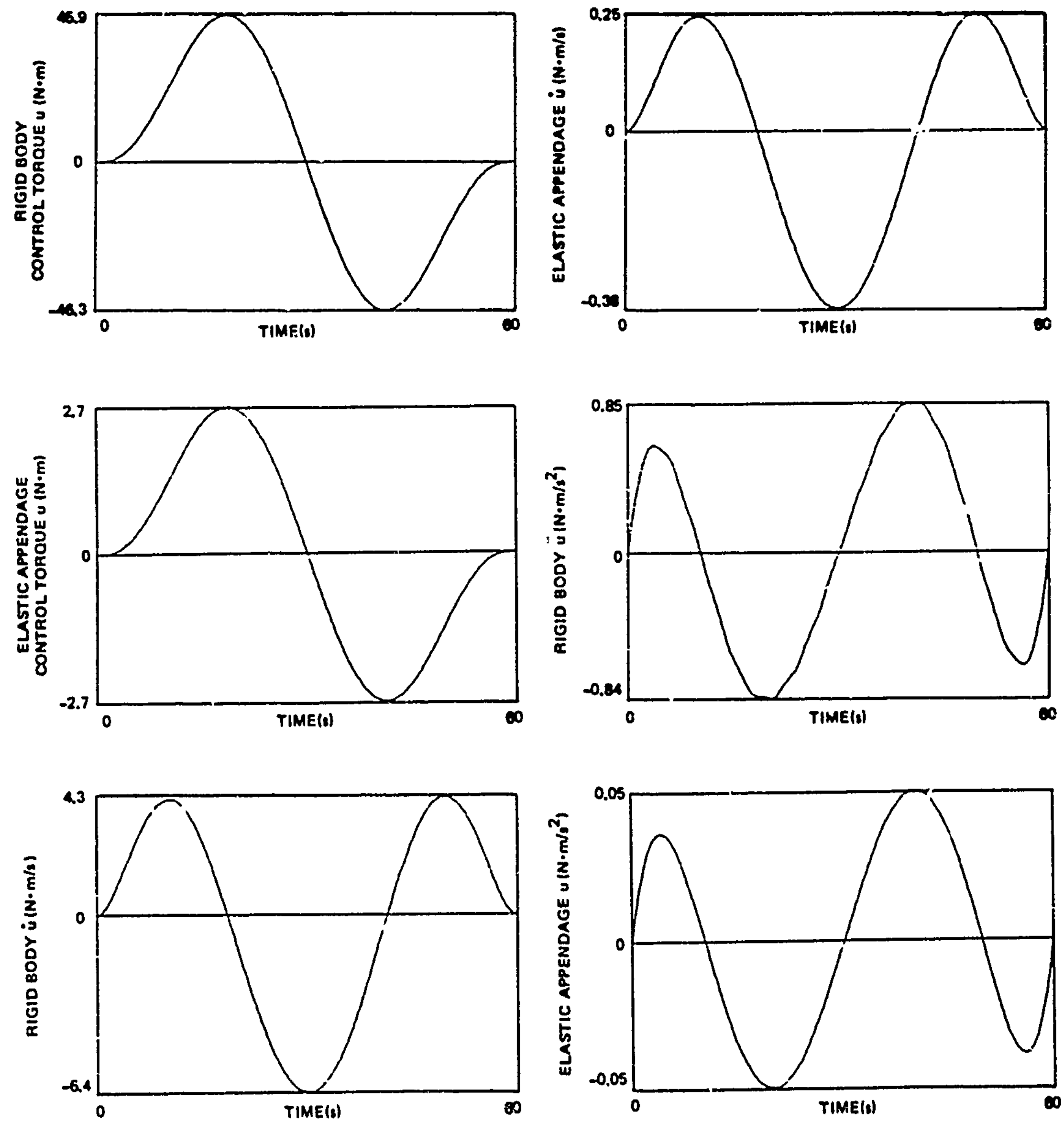
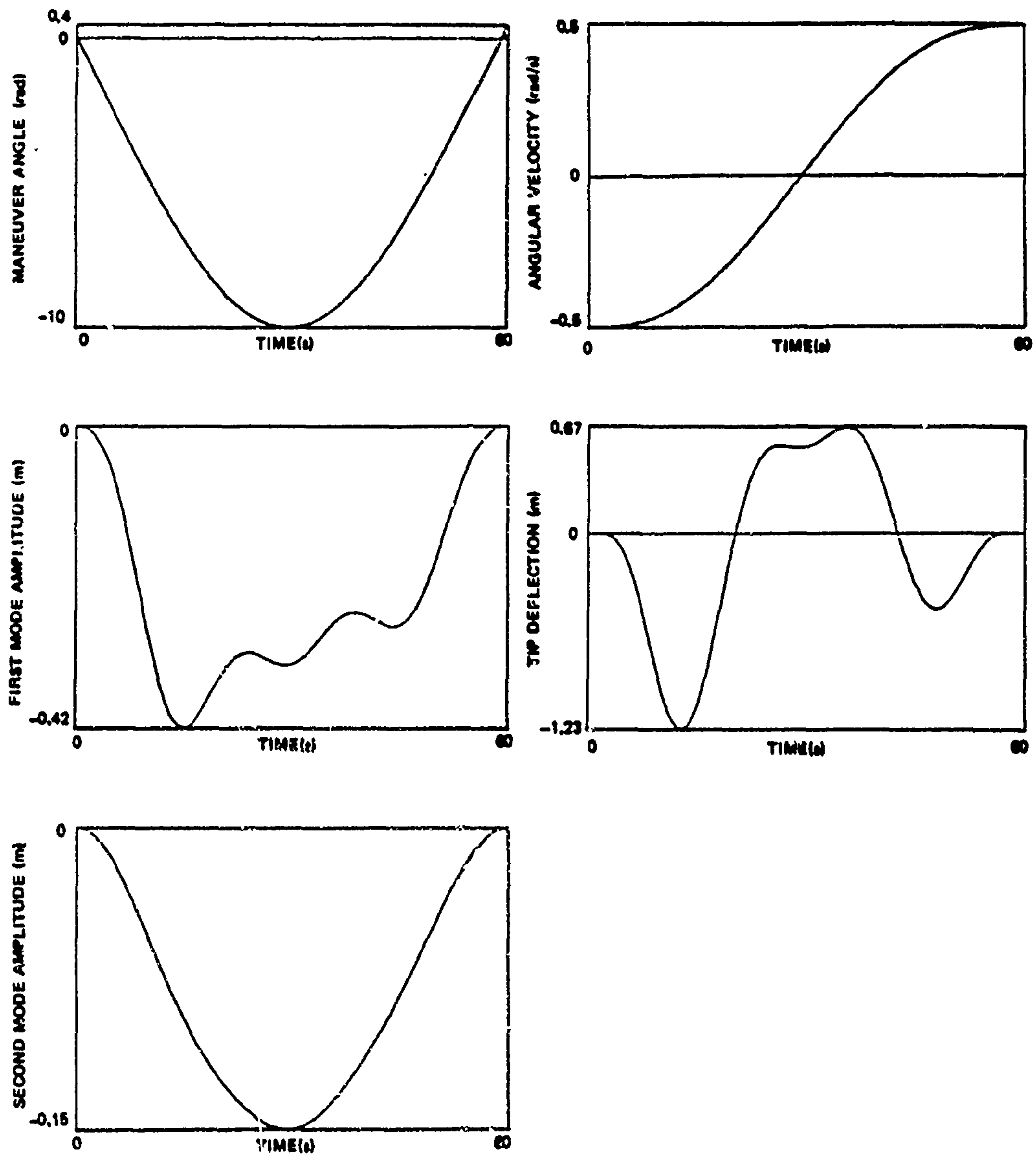


Figure 8-7. Control profiles for Case 3.



(a)

Figure 8-8. Case 4, free final angle rotation reversal maneuver with control rate penalties in the performance index.

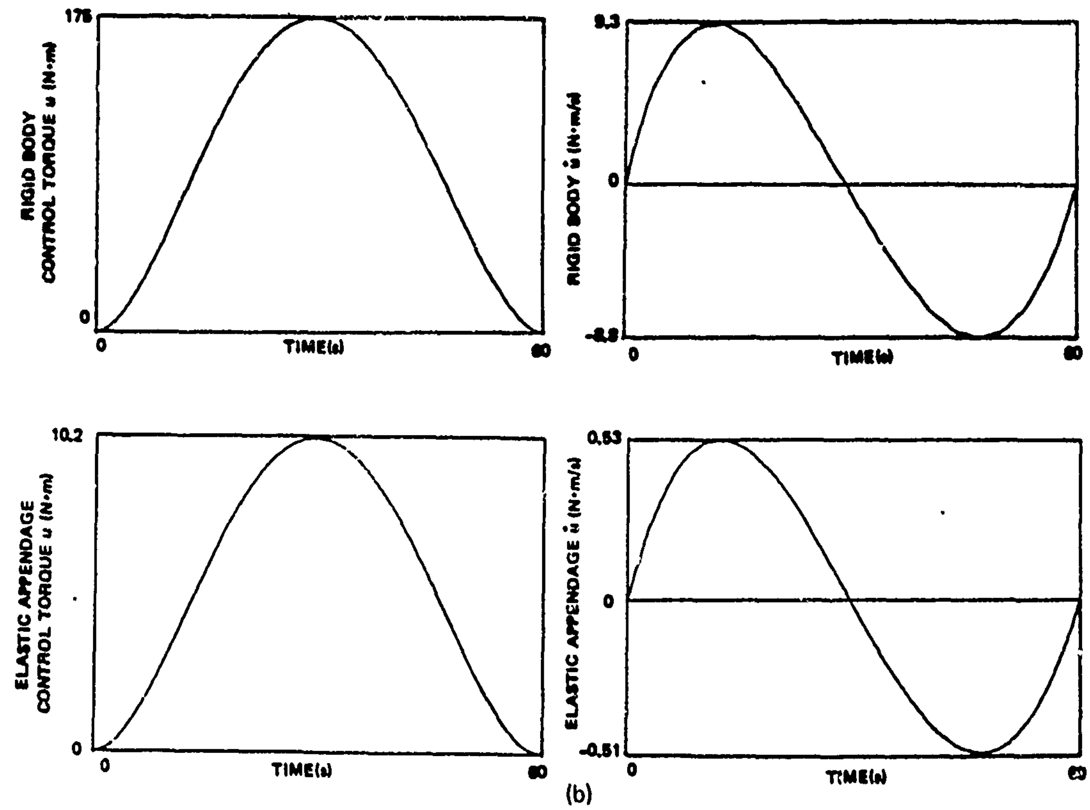


Figure 8-9. Control profiles for Case 4.

8.6.4 Case 5 (Figures 8-10 through 8-14)

Case 5 is a 5-second, 5-degree, rest-to-rest maneuver of the ACOSS Model 2 about the z axis. The maneuver is simulated for 7 seconds in order to observe the residual vibrations. The formulation of Section 8.3 is used with the performance index penalizing the state, control, and first and second time derivatives of the control. The first two antisymmetric modes, Modes 10 and 11, are controlled with the others uncontrolled. The results show that the symmetric modes are only slightly excited. Mode 7, which is a vibration in the x-z plane, and Mode 8, which is a symmetric vibration in the x-y plane, are excited only slightly. Modes 9, 12, and 18, which are mainly vibrations in the x-z plane with some antisymmetric components on the x-y plane, are excited somewhat more than Modes 7 and 8 because of the antisymmetri , component.

Modes 12 and 18 have small amplitude residual ringing at the end of the maneuver, but Mode 9 has a relatively large amplitude at the terminal time. Modes 10 and 11, which are the controlled antisymmetric modes, are excited but are forced to have essentially zero amplitude at the end of the maneuver, as expected. Mode 15, which is a twisting of the solar panels, is hardly excited. Mode 17, which is purely antisymmetric in the x-y plane, is excited, but the amplitude is greatly reduced at the terminal time with small amplitude residual ringing. The large residual amplitude of Mode 9 suggests that it should be controlled in future simulations.

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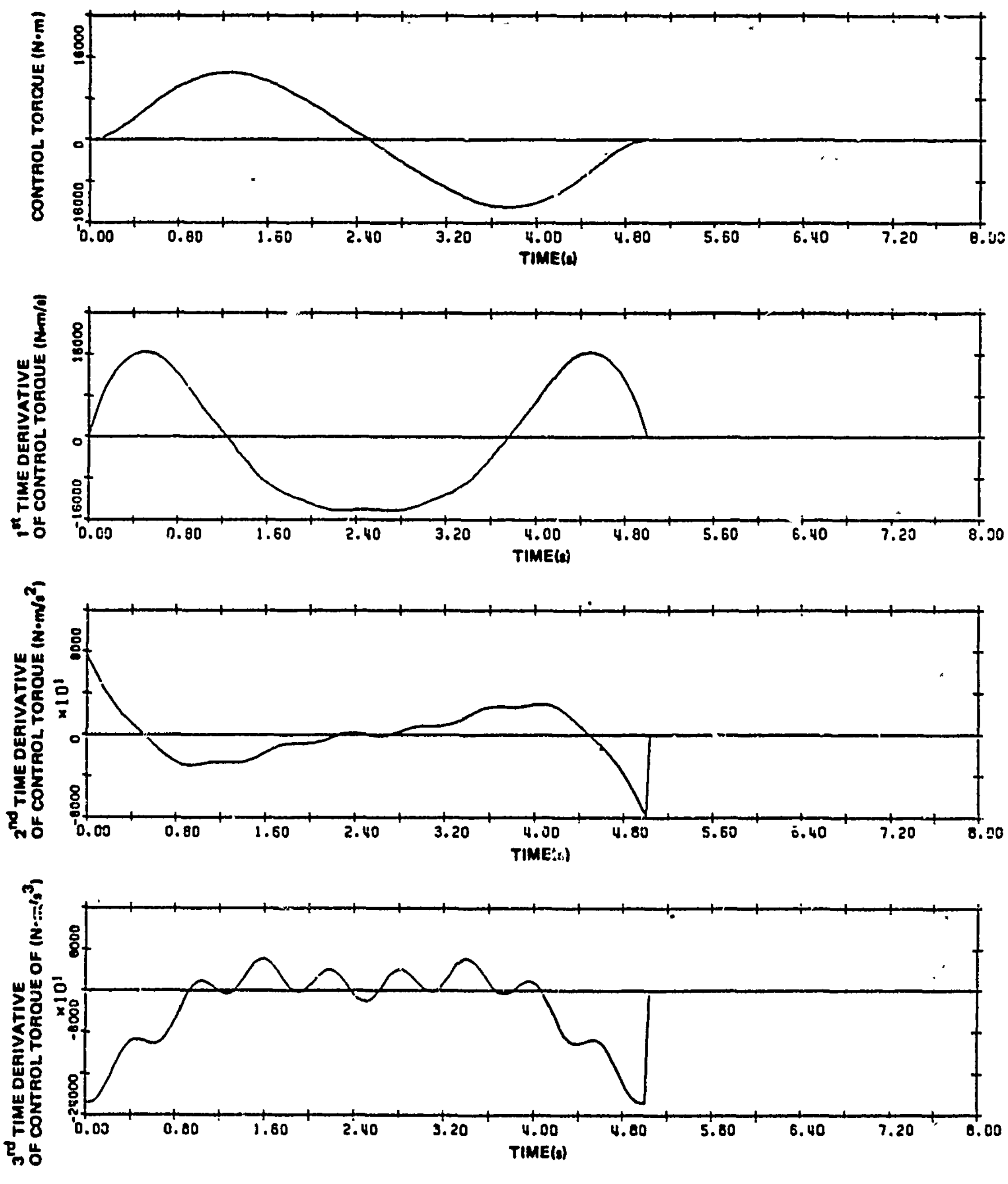

Figure 8-10. Control torque profile for Case 5.

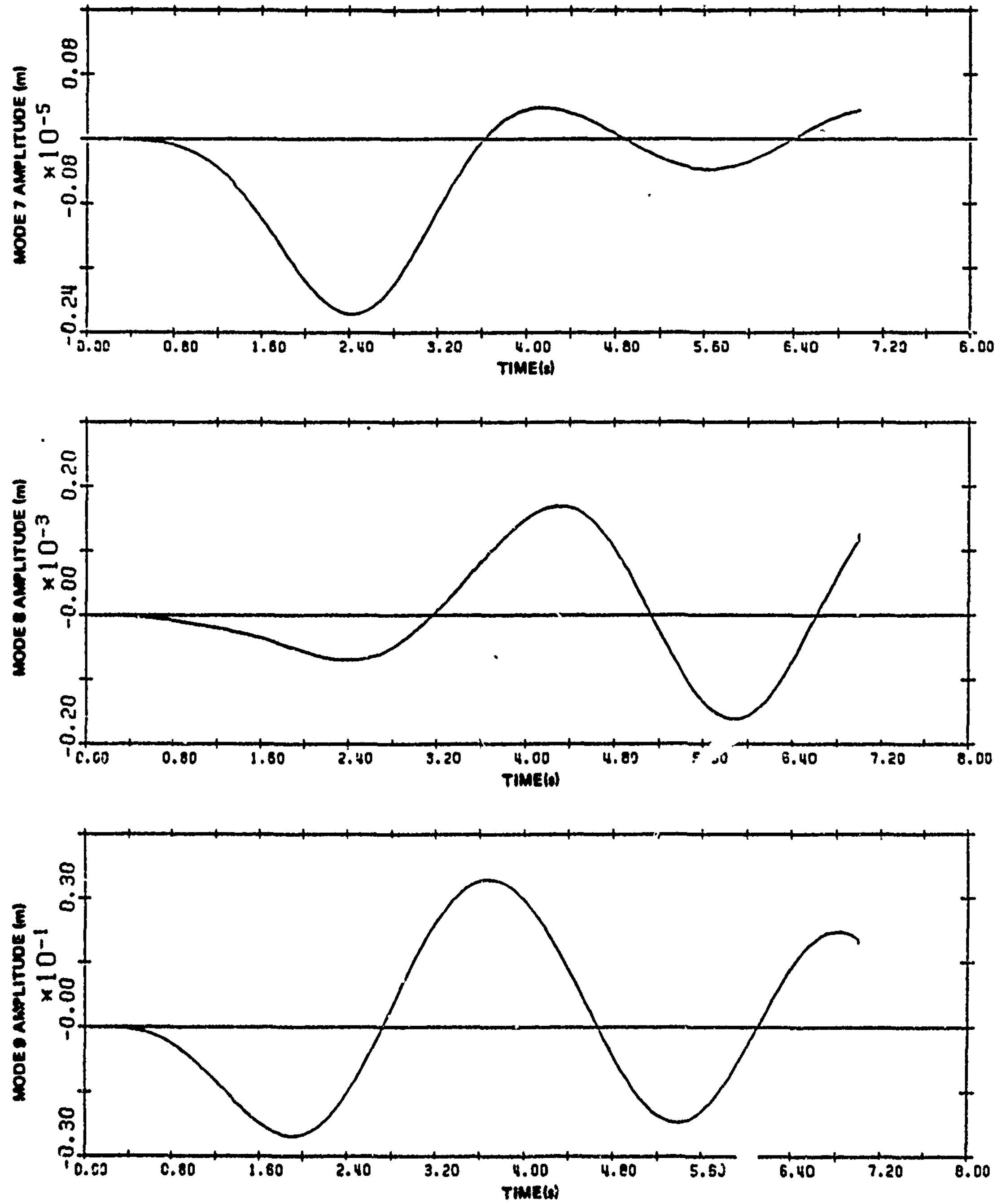

Figure 8-11. Case 5, time response of Modes 7, 8, and 9.

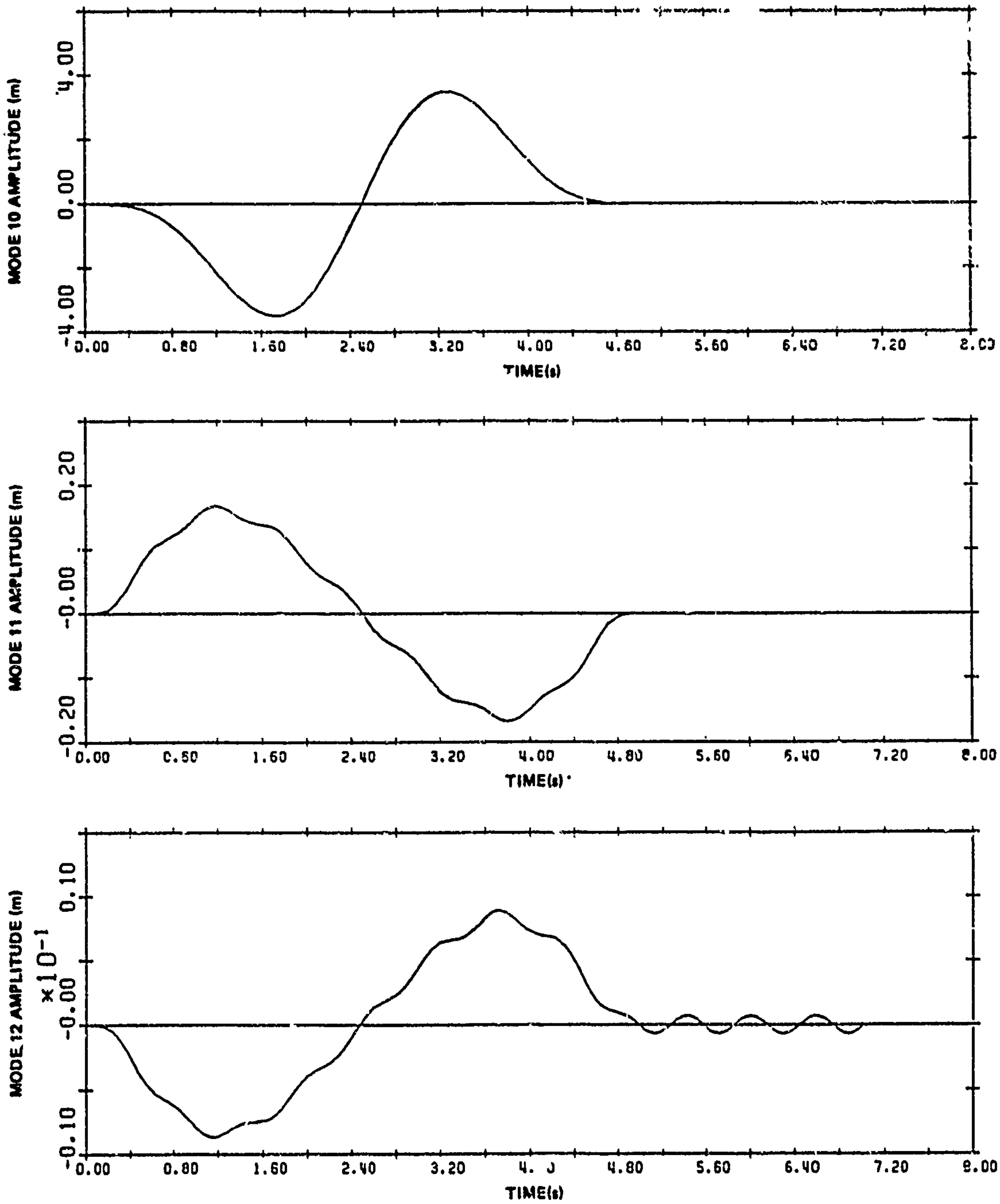

Figure 8-12. Case 5, time response of Modes 10, 11, and 12.

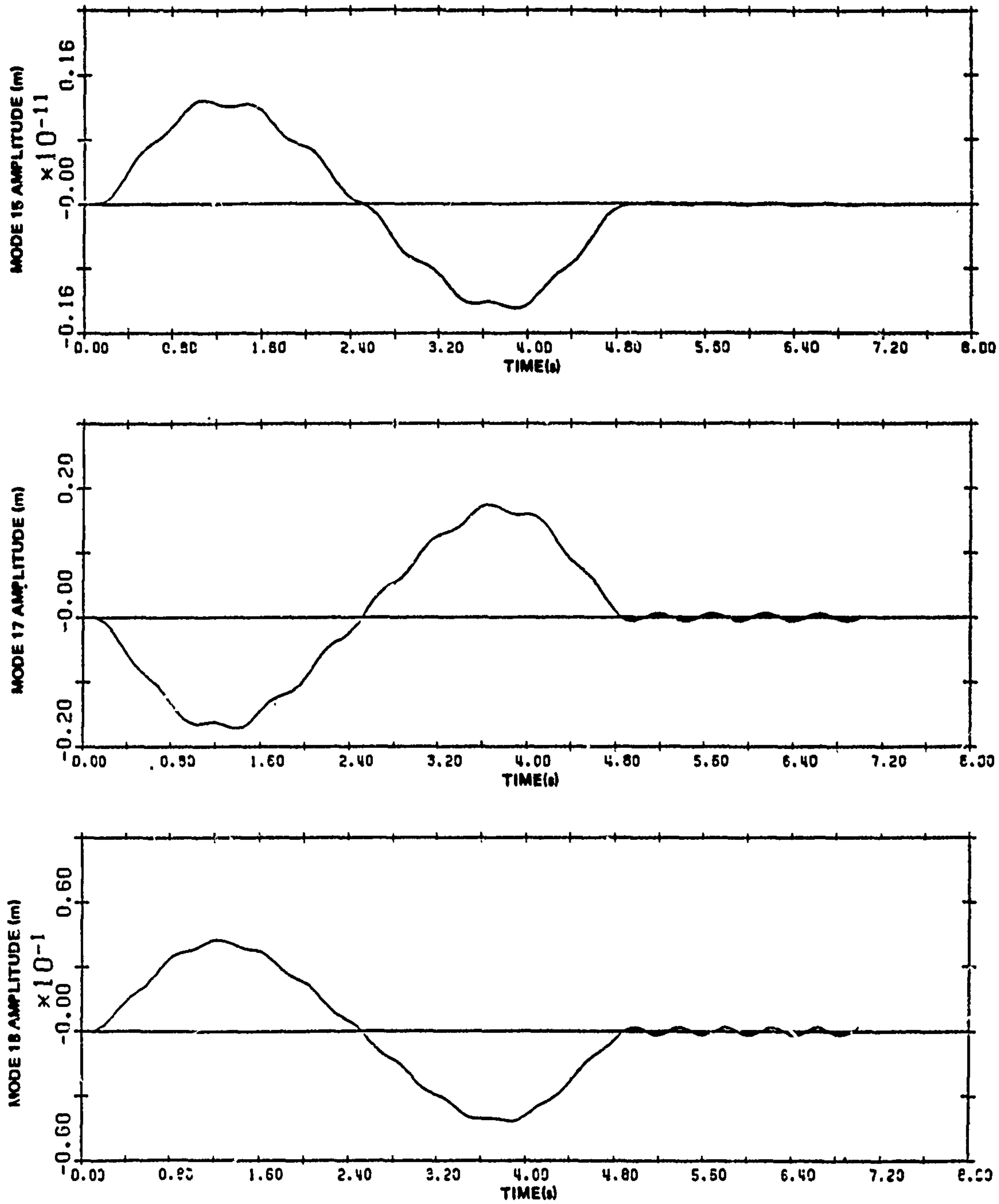

Figure 8-13. Case 5, time response of Modes 15, 17, and 18.

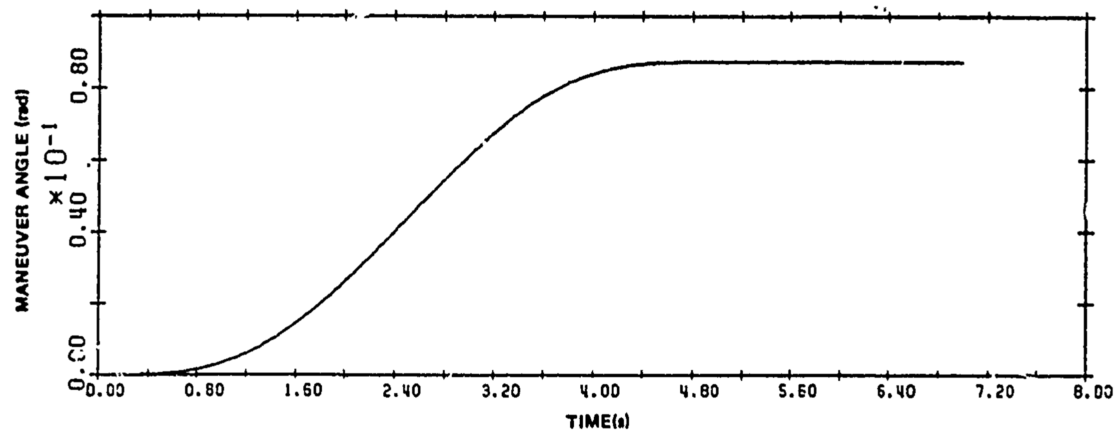

Figure 8-14. Case 5, body frame maneuver angle.

SECTION 9

ORDER REDUCTION BY IDENTIFICATION—SOME ANALYTICAL RESULTS

9.1 Introduction

Considerable attention in the community studying large space structures (LSS) has been devoted to the design of (stable) controllers. The major difficulties encountered are the dimensionality problem and the spillover problem. Various devices have been suggested to overcome these difficulties. Roughly speaking, the common philosophy of all the designs is: select n dominant modes (n relatively small), design a controller for the n modes while preventing (or minimizing) the spillover effect. Although this is a plausible philosophy, it suffers from a major flaw: the success of the corresponding design depends on knowledge of a precise high-order model. Thus, the burden of dealing with a high-order model is transformed from a control design problem to a modeling problem. Unfortunately, the identification of high-order models is even more imperilled with numerical difficulties than controlling such systems. Thus, it seems that a compromise between these difficulties is imminent. Actually, such a compromise is practiced regularly by engineers as follows:

(1) A model structure is selected (e.g., linear of order n).

(2) Using experimental data, the best model of the selected structure is estimated.

(3) A controller is designed for the "best model".

(4) The performance of the controller when applied to the plant is tested.

Although a common practice, no proof exists that such a scheme will work, except for Step (4). The purpose of this section is to evaluate the feasibility of this approach theoretically. Specifically, an attempt is made to characterize control designs that will guarantee stability despite the order reduction induced by the identification. Since the least squares (LS) method is one of the more robust identification schemes, and since analytical expressions for order reduction exist for the LS, this method is analyzed herein.

Section 9.2 discusses general properties of LS identification, especially those pertaining to order reduction. A general stability theorem for the discrete-time system and its relation to LS is presented subsequently.

Sections 9.3 and 9.4 characterize controllers which are stable when designed based on reduced-order models. It is shown that stability enhancing controllers (i.e., controllers which increase damping) are robust when applied to systems with reduced-order models. In the last section conclusions are drawn and extensions are outlined.

Nomenclature

All vectors are denoted by underlined lower case letters, e.g., $\underline{x}$, $\underline{a}_n$, $\underline{a}^n$.

The j^{th} entry of $\underline{x}$ is x_j.

Matrices and operators are denoted by capital letters, e.g., P_n, X, E, J

The dimensions of the matrices (operators) are not specified if they can be deduced from the context.

9.2 Least-Squares Identification

To facilitate the analysis of the subsequent section, we present first some basic results in LS theory.

Let $\{\underline{x}_0, \underline{x}_1, \ldots\}$ be a set of vectors in a Hilbert space H.

Let

$$L_n \triangleq \text{Span}\{\underline{x}_1, \ldots, \underline{x}_n\}$$

and

$$\hat{\underline{x}}_0^n \triangleq P_n \underline{x}_0 \triangleq \text{projection of } \underline{x}_0 \text{ on } L_n \tag{9-1}$$

$$= \sum_{i=1}^{n} \hat{a}_i^n \underline{x}_i$$

Then the following results:

Result 9-1

$$\hat{\underline{x}}_0^{n+1} = \sum_{i=1}^{n+1} \hat{a}_i^{n+1} \underline{x}_i \tag{9-2}$$

$$\hat{a}_i^{n+1} = \hat{a}_i^n - \hat{a}_{n+1}^{n+1} \alpha_{n+1,i} \ , \ i = 1,\ldots,n \tag{9-3}$$

where

$$P_n\underline{x}_{n+1} = \sum_{i=1}^{n} \alpha_{n+1,i}\underline{x}_i$$

and where $\hat{a}_{n+1}^{n+1}\ \bar{\underline{x}}_{n+1}$ is the projection of $\underline{x}_0$ on $\bar{\underline{x}}_{n+1}$; $\bar{\underline{x}}_{n+1} \triangleq \underline{x}_{n+1} - P_n\underline{x}_{n+1}$.

Result 9-1 is best illustrated as shown in Figure 9-1.

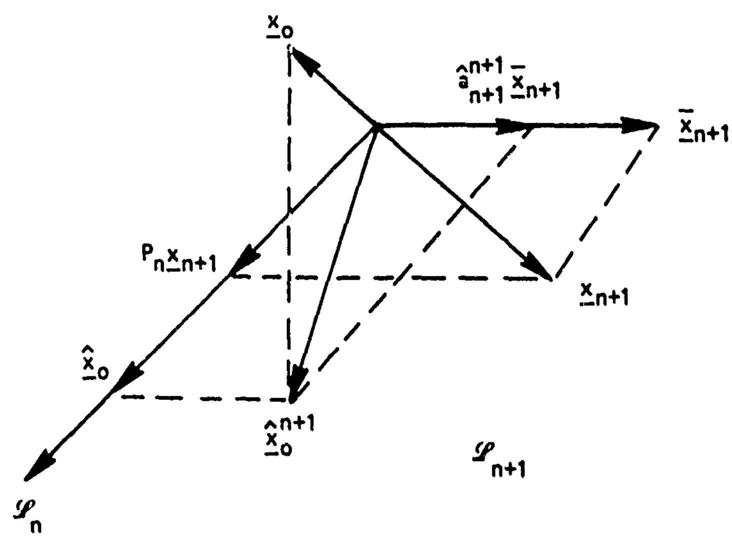

Figure 9-1.

The algebraic relations which correspond to Result 9-1 can be expressed as follows

$$P_n\underline{x}_0 = X_n(X_n^T X_n)^{-1} X_n^T\underline{x}_0 \tag{9-4}$$

where

X_n is the matrix

$$X_n = [\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n]$$

and the superscript T denotes the transpose operation. Defining the vector

$$\underline{\hat{a}}^n = \begin{bmatrix} a_1^n \\ \vdots \\ \hat{a}_n^n \end{bmatrix}$$

and the truncation operator E

$$E\underline{x} = E\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

we have

$$\hat{a}_{n+1}^{n+1} = \left(\bar{\underline{x}}_{n+1}^T \; \bar{\underline{x}}_{n+1}\right)^{-1} \bar{\underline{x}}_{n+1}^T \underline{x}_0 \tag{9-5}$$

$$E\underline{\hat{a}}^{n+1} = \underline{\hat{a}}_{n+1}^{n+1} \left(X_n^T X_n\right)^{-1} X_n^T \underline{x}_{n+1} \tag{9-6}$$

The application of Eq. (9-5) and (9-6) results in the system identification theory as follows. Let

$$y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=0}^{m} b_i u(k-1) + w(k) \tag{9-7}$$

be an (n,m) ARIMA model to be identified, where {y()}, {u()}, and {w()} are output, input, and noise (residuals) sequences, respectively. Define

$$\underline{x}_i = \begin{bmatrix} y(n-i) \\ \vdots \\ y(N-i) \end{bmatrix} \quad i = 0,\ldots,n$$

$$\underline{x}_i = \begin{bmatrix} u(2n+1-i) \\ u(N+n+1-i) \end{bmatrix} \quad i = n+1,\ldots,n+m$$

where N is the length of the data.

The LS estimate of the (n,m) ARIMA model can be transformed to the (n + 1,m) or the (n,m + 1) order ARIMA model using the relations above.

Assuming that the number of unstable poles is known and choosing n larger than that number, it can be shown [9-1] that the unstable modes can be identified exactly using the LS algorithm. Thus, in the sequel, only the effect of order reduction on stationary (stable) systems is considered.

In the stationary case, assume that N is large enough such that

$$\frac{1}{N}\sum_{k=\ell}^{N} y(k)y(k-\ell) \simeq E[y(k)y(k-\ell)] = v(\ell) \tag{9-8}$$

Thus, the LS identification of the (n,0) model is given by the last column of V_n^{-1} where

$$V_n = \text{Toep}\,[v(0),v(1),\ldots,v(n)]$$

$$\triangleq \begin{bmatrix} v(0) & v(1) & \cdots & v(n) \\ v(1) & v(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ v(n) & \cdots & & \end{bmatrix} \tag{9-9}$$

It follows from the Toeplitz structure of V_n that the relation in Result 9-1 can be written as

$$\underline{\hat{a}}^{n+1} = \begin{bmatrix} \underline{\hat{a}}^{n} - \hat{a}^{n+1}_{n+1} J\underline{\hat{a}}^{n} \\ \text{--------------} \\ \hat{a}^{n+1}_{n+1} \end{bmatrix} \tag{9-10}$$

where

$$J \triangleq \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}$$

i.e.,

$$J \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ \vdots \\ x_1 \end{bmatrix}$$

Equation (9-10) is the basis for the celebrated lattice structure of stable systems (see, e.g., [9-2]). Note that

$$\underline{\hat{a}}^{n} = \frac{1}{1 - (\hat{a}^{n+1}_{n+1})^2} \left[E\underline{\hat{a}}^{n+1} + \hat{a}^{n+1}_{n+1} JE\underline{\hat{a}}^{n+1}\right] \tag{9-11}$$

The intimate relation between V_n, $\underline{\hat{a}}^n$, and stability theory has been discussed in a number of articles [9-3, 9-4, 9-5] and is summarized in Theorem 9-1.

<u>Theorem 9-1</u>

Statements (1) through (5) regarding the polynomial

$$P_n(z) = z^n - \sum_{i=1}^{n} a_i z^{n-1}$$

and the corresponding companion matrix

$$A = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I & \\ 0 & & & \\ \hline a_n & \cdots & & a_1 \end{array}\right]$$

are equivalent.

(1) $P_n(z) = \det[zI - A]$ has its roots inside the unit circle.

(2) The Toeplitz matrix solution of the Ricatti equation

$$AV_nA^T + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [0 \ \ldots \ 0 \ 1] = V_n \tag{9-12}$$

is positive definite.

(3) M_n, defined in Eq. (9-13) is positive definite

$$M_n \triangleq \begin{bmatrix} 1 & & & 0 \\ -a_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ -a_n & \cdots & -a_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a_1 & \cdots & -a_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & -a_1 \\ 0 & & & 1 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & & 0 \\ -a_n & \ddots & & \\ \vdots & \ddots & \ddots & \\ -a_1 & \cdots & -a_n & 0 \end{bmatrix} \begin{bmatrix} 0 & -a_n & \cdots & -a_1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & -a_n \\ 0 & & & 0 \end{bmatrix}$$

$$= A_1A_1^T - A_2A_2^T \tag{9-13}$$

(4) The Schur-Cohn matrix SC_n, defined below, is positive definite

$$SC_n \triangleq \begin{bmatrix} 1 & & & 0 \\ -a_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ -a_{n-1} & \cdots & -a_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a_1 & \cdots & -a_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & -a_1 \\ 0 & & & 1 \end{bmatrix}$$

$$- \begin{bmatrix} -a_n & & 0 \\ \vdots & \ddots & \\ -a_1 & \cdots & -a_n \end{bmatrix} \begin{bmatrix} -a_n & \cdots & -a_1 \\ & \ddots & \vdots \\ 0 & & -a_n \end{bmatrix}$$

(5) Let $\hat{\underline{a}}^1, \hat{\underline{a}}^2, \ldots, \hat{\underline{a}}^n$ be the LS parameter vectors of the first to the n^{th} order AR models obtained from the stationary data $\{y(i)\}$. then $|\hat{a}_i^i| < 1$, $i = 1,\ldots,n$.

(6) Let $\underline{a}^n, \underline{a}^{n-1}, \ldots, \underline{a}^1$ be a set of vectors obtained using Eq. (9-10) and (9-11) repeatedly. Then, $|a_i^i| < 1$, $i = 1,\ldots,n$.

Theorem 9-1 shall serve in assessing the effect of order reduction on the stability of a feedback design.

Note in particular that if $P_n(z)$ is stable, V_n of Statement 2 and M_n^{-1} of Statement 3 are identical to the covariance matrix of the (n,0) ARIMA model excited by unit variance white noise. The main benefits in the stability statements of the theorem above are that they tie stability to model reduction and LS estimation and that explicit expressions (the matrices M_n or V_n) to be used in stability analysis are given in terms of the model parameters (as opposed to the implicit Lyapunov conditions). These features will be exploited in the following analysis.

9.3 Second-Order System

In this section, we shall consider the effect of designing a stable feedback for a second-order plant when the designer assumes a model of order one. This simple case will give some insight into the more general case to be addressed in Section 9.4.

Consider the stable (2,1) system

$$y(k) = a_1^2 y(k-1) + a_2^2 y(k-2) + u(k) \tag{9-14}$$

modeled by the first-order model (obtained through LS identification)

$$y(k) = a_1^1 y(k - 1) + u(k) \tag{9-15}$$

By the previous discussion

$$a_1^1 = \frac{a_1^2}{1 - a_2^2}$$

and $|a_1^1| < 1$, $|a_2^2| < 1$ by assumption. Consider the feedback

$$u(k) = -pa_1^1 y(k - 1)$$

A stable design restricts p to satisfy

$$|1 - p| < \frac{1}{\text{[illegible]}} \tag{9-16}$$

However, to guarantee closed-loop stability of the plant, we must satisfy

$$\left| \frac{a_2^2 - pa_1^1}{1 - a_2^2} \right| < 1$$

or equivalently

$$|1 - a_2^2 - p| < \frac{1 - a_2^2}{|a_1^1|} \tag{9-17}$$

The constraints, Eq. (9-16) and (9-17), are plotted in Figure 9-2.

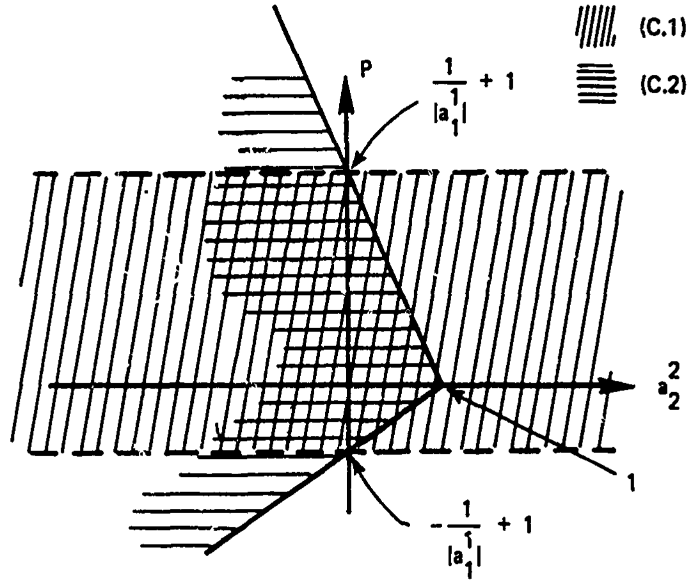

Figure 9-2.

It follows that for $a_2^2 < 0$, any control law that satisfies Eq. (9-16) is stable when applied to the full-order plant. If $a_2^2 > 0$, there exist no $p \neq 0$ for which stability can be guaranteed. Note that if λ_1, λ_2 are the plant eigenvalues, $a_2^2 = -\lambda_1\lambda_2$. Thus, $a_2^2 > 0$ only if $\lambda_1\lambda_2 < 0$, which is possible only if both eigenvalues are real and sign $(\lambda_1) = -\text{sign}(\lambda_2)$. The failure to guarantee stability using a reduced order model of such a system is not surprising. We note that the result above is also applicable in the case of a second-order system with one unstable mode, say λ_1^1, and where $a_1^1 = \lambda_1$. This discussion is summarized in Lemma 9-1.

<u>Lemma 9-1</u>

Let Eq. (9-14) be the system equation and Eq. (9-15) be a model obtained from Eq. (9-14) by LS definition. Then, the feedback $u(k) = \beta y(k - 1)$, satisfying $|a_1^1 + \beta| < 1$, is stable if $a_2^2 < 0$.

Conversely, there always exists $a_2^2 > 0$ such that this feedback is unstable.

Note in particular that for $-1 > a_2^2 < 0$ and $|a_1^1| < 1$, $0 < p < 2$ guarantees stability, and that $|a_1^1(1 - p)| < |a_1^1|$ in this case. We shall refer to such a control law as stability enhancing control (see Definition 9-2 in Section 9.4).

It is reasonable to assume that the analysis of the residual energy will provide tighter bounds on $|a_2^2|$ thus allowing for a more judicial choice of p to accommodate $a_2^2 > 0$. In particular, note that for $0 < p < 1$, stability is guaranteed if $-1 < a_2^2 < \dfrac{1 + (1 - p)|a_1^1|}{1 + |a_1^1|}$.

9.4 n^{th} Order Systems

In this section, the results of the previous section are generalized. To enable this generalization, a few definitions and relations must first be presented.

Definition 9-1:

$$S \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ & & 0 \\ & I & \vdots \\ & & 0 \end{bmatrix} \tag{9-18}$$

S is called the shift operator since

$$S \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} .$$

Notation

The vector of coefficients in the polynomial $P_n(z)$ of Theorem 9-1 will be denoted by $\underline{\alpha}_n$, i.e.,

$$\underline{\alpha}_n^T \triangleq [1,-a_1,\ldots,-a_n] \tag{9-19}$$

It follows that the matrices A_1 and A_2 defined in Eq. (9-13) satisfy

$$A_1 = [\underline{\alpha}_n \quad S\underline{\alpha}_n \quad S^n\underline{\alpha}_n] \tag{9-20}$$

$$A_2 = S[J\underline{\alpha}_n \quad SJ\underline{\alpha}_n \quad \ldots \quad S^nJ\underline{\alpha}_n] \tag{9-21}$$

or equivalently, from Eq. (9-13)

$$M_n = \sum_{i=0}^{n} [(S^i\underline{\alpha}_n)(S^i\underline{\alpha}_n)^T - (S^{i+1}J\underline{\alpha}_n)(S^{i+1}J\underline{\alpha}_n)^T] \tag{9-22}$$

By Theorem 9-1, $P(z)$ is stable if and only if $M_n > 0$, i.e.

$$\underline{x}^T M_n \underline{x} = \sum_{i=0}^{n} [(\underline{x}^T S^i \underline{\alpha}_n)^2 - (x^T S^{i+1} J\underline{\alpha}_n)^2]$$

$$> 0 \quad \forall\, \underline{x} \epsilon R^{n+1} \tag{9-23}$$

Let $\underline{\alpha}_{n+1}$ be related to $\underline{\alpha}_n$ via Eq. (9-10) and (9-11), and define

$$\bar{\underline{x}} = \begin{bmatrix} y \\ --- \\ \underline{x} \end{bmatrix} \epsilon R^{n+2}, \quad \underline{x} \epsilon R^{n+1}$$

Then we obtain

$$\bar{\underline{x}}^T M_{n+1} \bar{\underline{x}} = \left[1 - (a_{n+1}^{n+1})^2 \quad \underline{y}^T M_n \underline{x}\right]$$

$$+ \left(\bar{\underline{x}}^T \begin{bmatrix} \underline{\alpha}_n \\ 0 \end{bmatrix} - a_{n+1}^{n+1} \bar{\underline{x}}^T M_n \begin{bmatrix} \underline{\alpha}_n \\ 0 \end{bmatrix} \right)^2 \tag{9-24}$$

Thus, $M_{n+1} > 0$ if $M_n > 0$ and $|a_{n+1}^{n+1}| < 1$ as in Theorem 9-1. Finally, the following concept is needed for the generalization of Section 9.3.

Let $V_n = \text{Toep}[v_0, \ldots, v_{n+1}]$ be the solution of Eq. (9-12) corresponding to $\underline{\alpha}_n$. Consider the stable feedback law

$$u(k) = \underline{\beta}_n^T \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-n) \end{bmatrix} \qquad (9\text{-}25)$$

and define

$$\bar{\underline{\alpha}}_n \triangleq [1 \;\vdots\; -(a_1^n + \beta_1) \;\vdots\; , \ldots, \; -(a_n^n + \beta_n)]^T$$

and correspondingly, define $\bar{V}_n$, $\bar{P}_n(z)$, etc.

<u>Definition 9-2</u>: $\underline{\beta}_n$ is said to be a <u>stability enhancing control</u> law if $\bar{V}_n \leq V_n$ or equivalently $\bar{M}_n \geq M_n$.

Using the concept in Definition 9-2, the main result of this study is obtained.

<u>Theorem 9-2</u>

Let $\underline{\alpha}_{n+1}$ and $\underline{\alpha}_n$ be related via Eq. (9-10) and (9-11). Let $\underline{\beta}_n$ be a stability enhancing control, designed for the system $\underline{\alpha}_n$. Then the control law, Eq. (9-25), is stable when applied to the system $\underline{\alpha}_{n+1}^{n+1}$ if it satisfies $\forall\, \underline{x} \in R$

$$\begin{aligned}(1 - (a_{n+1}^{n+1})^2)\, \underline{x}^T M_n \underline{x} + 2a_{n+1}^{n+1} \sum_{i=0}^{n} [(\underline{x}^T S^{i+1} J\bar{\underline{\alpha}}_n)(\underline{x}^T S^i \underline{\alpha}_n) \\ - (\underline{x}^T S^i \bar{\underline{\alpha}}_n)(\underline{x}^T S^{i+1} J\underline{\alpha}_n)] \\ = x^T [(1 - (a_{n+1}^{n+1})^2) M_n + a_{n+1}^{n+1} R_n]\, \underline{x} \geq 0 \qquad (9\text{-}26)\end{aligned}$$

Remarks

(1) $\underline{\bar{a}}_n$ in Eq. (9-26) can be replaced by $\begin{bmatrix} 0 \\ \underline{\beta}_n \end{bmatrix}$.

(2) The matrix R_n is, in general, nondefinite. However, the sign definiteness of $\left[1 - \left(a_{n+1}^{n+1}\right)^2\right] M_n + a_{n+1}^{n+1} R_n$ as a function of the value of a_{n+1}^{n+1} can be established for a given choice of stability enhancing control.

(3) For $n = 1$, it is easily verified that Eq. (9-26) implies $-1 < a_2^2 < 0$, which is consistent with Lemma 9-1.

(4) Note that $\left[1 - \left(a_{n+1}^{n+1}\right)^2\right] M_n + a_{n+1}^{n+1} R_n \to M_n$ as $a_{n+1}^{n+1} \to 0$, which is consistent with reasoning.

(5) The proof of the theorem is obtained by using Eq. (9-10), (9-11), and (9-12); the notion of stability enhancing control and using the identities

$$\underline{\bar{x}}^T S = [\underline{x}^T 0] \; ; \quad \underline{\bar{x}}^T S^i J \begin{bmatrix} \underline{\alpha}_n \\ 0 \end{bmatrix} = \underline{x}^T S^i J \underline{\alpha}_n$$

9.5 Conclusions

The effect of order reduction on the stability of feedback design has been addressed. The assumption made was that the reduction of order is based on the results of system identification rather than on analytical reduction of known full-order systems.

The stability analysis was based on the relation between LS estimation and stability theory. Sufficient conditions for a "stability enhancing controller" to be stable when applied to a full-order system have been established.

This chapter presents only a first stab at understanding the effect of model reduction induced by identification on stability. The generalization of Theorem 9-2 to an arbitrary difference in orders between the model and the plant yields similar conditions to that of Eq. (9-26). However, explicit constraints on the ignored parameters are cluttered by the algebra. Further studies are necessary to alleviate this algebraic problem.

In the discussion, it was assumed that the ignored parameter a_{n+1}^{n+1} is absolutely bounded by one. Actually, one should expect that, by choosing n sufficiently large, tighter bounds on a_{n+1}^{n+1} can be found (e.g., by relating a_{n+1}^{n+1} to the energy in the residual sequence). Such an analysis in the time domain is analogous to the often used assumption of poor knowledge of the frequency response of a plant in the high-frequency range (see e.g., [9-6]).

The results presented in this section are the first to give theoretical justification to a common practice. Success of simplified modeling has been reported in various fields (e.g., power generation, which is very similar to LSS in its characteristics [9-7]). It is our intention to demonstrate the practicality of this approach on Draper Model #2 in the near future.

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